From Wavelets to Shearlets and back again

Gitta Kutyniok and Tomas Sauer

Abstract. In this paper we will study the Continuous Shearlet Transform from a wavelet point of view, and show how this perspective can be used to derive a new geometric interpretation of this transform providing the possibility for FFT-based fast methods to compute the Continuous Shearlet Transform.

§1. Introduction

One main focus of current research is on the development of sparse directional representation systems, which not only provide the means to detect orientations of objects in the data such as curvilinear singularities, but also to lead to sparse representations, i.e., with most coefficients of the expansion being close to zero. In the past, several new representation systems were proposed, including the directional wavelets [1], the complex wavelets [11], the ridgelets [2], the contourlets [7], and the curvelets [3].

A very recent approach are shearlets, which not only possess all above mentioned properties, but are moreover equipped with a rich mathematical structure similar to wavelets. In fact, we will point out that shearlets can be seen as an appropriate shear and inverse shear operator applied to an anisotropic wavelet transform, so that theory and algorithms from the Continuous Wavelet Transform can be carried over. In other words, while still being conceptually close to wavelets, shearlets offer a flexible enough extension to precisely detect the position and orientation of singularities [12] and to provide optimally sparse representations [10]. For more and up–to–date information on this rapidly expanding area, we refer the interested reader to [16].

In this paper we will focus on the Continuous Shearlet Transform. Since both transformations, the Continuous Shearlet Transform as well as the Continuous Wavelet Transform, can be understood as specializations of the general concept of affine systems, it will be illustrating to study the

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precise relation. Therefore, after having provided an elaborate introduction to the Continuous Shearlet Transform from a wavelet point of view in Section 2, the second part of the paper (Section 3) is concerned with "returning back to wavelets" in the sense of expressing the Continuous Shearlet Transform in terms of an anisotropic Continuous Wavelet Transform. We will also briefly discuss how this idea can be used to generate FFT–based fast methods for computing the Continuous Shearlet Transform.

§2. From Wavelets to Shearlets

The purpose of this section is to provide an elaborate introduction to the Continuous Shearlet Transform and to regard it from a wavelet point of view. For more details on the 2-dimensional Continuous Wavelet Transform we refer to [6], whereas information on the Continuous Shearlet Transform can be found in [12].

2.1. Continuous Wavelet Transform in 2-D

The 2-dimensional Continuous Wavelet Transform is based on affine systems generated by one single function $\psi \in L^2(\mathbb{R}^2)$ by means of

$$\{\psi_{At} = T_t D_A \psi = |\det A|^{-\frac{1}{2}}\, \psi(A^{-1}(\,\cdot\, - t)) : A \in G,\ t \in \mathbb{R}^2\},$$

where G is a subgroup of the regular 2×2 matrices, $GL(2,\mathbb{R})$, and we use $T_t f(x) = f(x - t)$, $t \in \mathbb{R}^2$ and $D_M f(x) = |\det M|^{-\frac{1}{2}} f(M^{-1}x)$, $M \in GL(2,\mathbb{R})$, to denote the *translation* and *dilation operator* on $L^2(\mathbb{R}^2)$, respectively. Recall that the wavelet system $\{\psi_{At}\}_{A,t}$ can also be interpreted as being generated by a special group representation. Let $\mathbb{A} = GL(2,\mathbb{R}) \times \mathbb{R}^2$ be the *affine group* endowed with multiplication given by $(A,t)(A',t') = (AA', t + At')$, and let $\pi : \mathbb{A} \to \mathcal{U}(L^2(\mathbb{R}^2))$ denote the representation of $\mathbb{A}$ defined by $\pi(A,t)\psi(x) = |\det A|^{-1/2}\psi(A^{-1}(x-t))$. This allows us to write the elements of a wavelet system as

$$\psi_{At} = \pi(A,t)\psi.$$

Let $A \in GL(2,\mathbb{R})$ be an *expanding matrix*, i.e., all eigenvalues of A are larger than one in modulus. Since the inverse of such a matrix is contractive, the family $\{M_a = A^{-\log a} : a > 1\}$, is a subgroup of contractions of $GL(2,\mathbb{R})$. Recall that the canonical choice for A is λI, for some $\lambda > 1$, and therefore $M_a = \operatorname{diag}\left(a^{-\log\lambda}\right)$. The *Continuous Wavelet Transform* $\mathcal{W}_\psi f : \mathbb{R}^+ \times \mathbb{R}^2 \to \mathbb{C}$ of some $f \in L^2(\mathbb{R}^2)$ is defined as

$$\mathcal{W}_\psi f(a,t) = \langle f, \psi_{at}\rangle = |\det M_a|^{-\frac{1}{2}} \left\langle f, \psi\left(M_a^{-1}(\cdot - t)\right)\right\rangle.$$

Provided ψ is chosen to be a *wavelet*, i.e., ψ satisfies the *admissibility condition*

$$\int_{\mathbb{R}} \frac{|\hat{\psi}(\omega)|^2}{|\omega|}\, d\omega < \infty,$$

each function $f \in L^2(\mathbb{R})$ can be reconstructed from its wavelet coefficients $\{\langle f, \psi_{at}\rangle : (a,t) \in \mathbb{R}^+ \times \mathbb{R}^2\}$.

2.2. Continuous Shearlet Transform

Let us now turn our attention to shearlets. Having recalled the definition and basic properties of the 2-dimensional Continuous Wavelet Transform in the subsection before, it will become evident that there are similarities in spirit between this transform and the Continuous Shearlet Transform.

The Continuous Shearlet Transform is also based on special affine systems generated by one single function $\psi \in L^2(\mathbb{R}^2)$, but the dilation matrices are now designed for detecting directions. For each $a > 0$ and $s \in \mathbb{R}$, let A_a denote the *parabolic scaling matrix* and S_s denote the *shear matrix* of the form

$$A_a = \begin{pmatrix} a & 0 \\ 0 & \sqrt{a} \end{pmatrix} \quad \text{and} \quad S_s = \begin{pmatrix} 1 & s \\ 0 & 1 \end{pmatrix},$$

respectively. Then the *(continuous) shearlet system* generated by $\psi \in L^2(\mathbb{R}^2)$ is defined by

$$\{\psi_{ast} = T_t D_{S_s A_a} \psi = a^{-\frac{3}{4}} \psi(A_a^{-1} S_s^{-1}(\,\cdot\, - t)) : a \in \mathbb{R}^+,\ s \in \mathbb{R},\ t \in \mathbb{R}^2\},$$

and the associated *Continuous Shearlet Transform* of some $f \in L^2(\mathbb{R}^2)$ is given by

$$\mathcal{SH}_\psi f : \mathbb{R}^+ \times \mathbb{R} \times \mathbb{R}^2 \to \mathbb{C}, \quad \mathcal{SH}_\psi f(a,s,t) = \langle f, \psi_{ast}\rangle.$$

A function $\psi \in L^2(\mathbb{R}^2)$ is called a *continuous shearlet*, if it satisfies the *admissibility condition*

$$\int_{\mathbb{R}^2} \frac{|\hat{\psi}(\xi_1,\xi_2)|^2}{\xi_1^2}\, d\xi < \infty.$$

In this case, each function $f \in L^2(\mathbb{R}^2)$ can be reconstructed from its shearlet coefficients $\{\langle f, \psi_{ast}\rangle : (a,s,t) \in \mathbb{R}^+ \times \mathbb{R} \times \mathbb{R}^2\}$.

We wish to mention that the shearlet systems can also be viewed from a group theoretic point of view. The associated locally compact group – the so-called *Shearlet group* $\mathbb{S}$ – is defined to be the set $\mathbb{R}^+ \times \mathbb{R} \times \mathbb{R}^2$ endowed with the multiplication

$$(a,s,t) \cdot (a',s',t') = (aa', s + s'\sqrt{a}, t + S_s A_a t').$$

Notice that this group is isomorphic to the semi-direct product $G \ltimes \mathbb{R}^2$ with G being defined by $G = \{S_s A_a : a \in \mathbb{R}^+, s \in \mathbb{R}\}$. Letting $\sigma : \mathbb{S} \to \mathcal{U}(L^2(\mathbb{R}^2))$ be the unitary representation of this group given by

$$\sigma(a,s,t)\psi(x) = a^{-\frac{3}{4}}\psi(A_a^{-1}S_s^{-1}(x-t)),$$

the link with shearlet systems is established by the relation

$$\psi_{ast} = \sigma(a,s,t)\psi.$$

2.3. Directionality

In [12] it was proven that the directions of singularities in a distribution f can be detected by considering the decay of the associated Continuous Shearlet Transform. The shearlet coefficients $\langle f, \psi_{ast}\rangle$ always decay rapidly, i.e., faster than any polynomial, as $a \to 0$ *except* when t is on the singularity and s points in the direction perpendicular to the singularity. In this section we will illustrate this behavior also by means of figures.

For our analysis – as in [12] – we choose a particular continuous shearlet ψ. Intuitively, the shearlet itself should already be stretched in one direction. The main ingredient in the definition of ψ are two univariate functions $\psi_1, \psi_2 \in L^2(\mathbb{R})$ such that

(C1) ψ_1 is a continuous wavelet, $\hat{\psi}_1 \in C^\infty(\mathbb{R})$, and supp $\hat{\psi}_1 \subseteq [-2, -\frac{1}{2}] \cup [\frac{1}{2}, 2]$, and

(C2) ψ_2 is such that $\hat{\psi}_2 \in C^\infty(\mathbb{R})$ and supp $\hat{\psi}_2 \subseteq [-1, 1]$.

Then we define $\psi \in L^2(\mathbb{R}^2)$ almost as a tensor product by

$$\hat{\psi}(\xi) = \hat{\psi}(\xi_1, \xi_2) = \hat{\psi}_1(\xi_1)\,\hat{\psi}_2(\tfrac{\xi_2}{\xi_1}). \tag{1}$$

Here the quotient $\frac{\xi_2}{\xi_1}$ is natural for the shear operation. The computation

$$\begin{aligned}\int_{\mathbb{R}^2} \frac{|\hat{\psi}(\xi_1,\xi_2)|^2}{\xi_1^2}\,d\xi &= \int_{\mathbb{R}} \frac{|\hat{\psi}_1(\xi_1)|^2}{\xi_1^2} \int_{\mathbb{R}} |\hat{\psi}_2(\tfrac{\xi_2}{\xi_1})|^2\,d\xi_2\,d\xi_1 \\ &= \int_{\mathbb{R}} \frac{|\hat{\psi}_1(\xi_1)|^2}{\xi_1} \int_{\mathbb{R}} |\hat{\psi}_2(\eta)|^2\,d\eta\,d\xi_1 < \infty\end{aligned}$$

now proves that ψ is indeed a continuous shearlet.

To illustrate directionality of the analyzing functions ψ_{ast}, we choose ψ_1 and ψ_2 as follows. Let b_n, $n > 0$, denote the nth cardinal B-spline, defined recursively by $b_1 = \chi_{[0,1)}$ and $b_n = b_{n-1} * \chi_{[0,1)}$ for $n > 1$. Then we define ψ_1 and ψ_2 by

$$\hat{\psi}_1(\omega) = b_2(\tfrac{4}{3}(-\omega - \tfrac{1}{2})) + b_2(\tfrac{4}{3}(\omega - \tfrac{1}{2})) \tag{2}$$

and

$$\hat{\psi}_2(\omega) = b_8(4(\omega + 1)). \tag{3}$$

(C1) and (C2) are satisfied, except for the infinite differentiability, which we give up for the benefit of fast implementation.

Figures 1(c) and 1(d) show the function $\psi_{0.3,0,0}$ in frequency domain. We clearly see that it is stretched in x-direction. In the time domain (Figures 1(a) and 1(b)), this function is stretched in y-direction, which is the direction perpendicular to the x-direction. The function $\psi_{0.3,-0.5,0}$ with shear parameter equal to $-\frac{1}{2}$ is illustrated in Figure 2. Images (c) and (d) can clearly be seen as a sheared version of Figures 1(c) and 1(d), and images (a) and (b) show this function in time domain. Again we notice that the direction, in which $\psi_{0.3,-0.5,0}$ is stretched, it precisely the direction perpendicular to the direction, in which $\widehat{\psi_{0.3,-0.5,0}}$ is stretched.

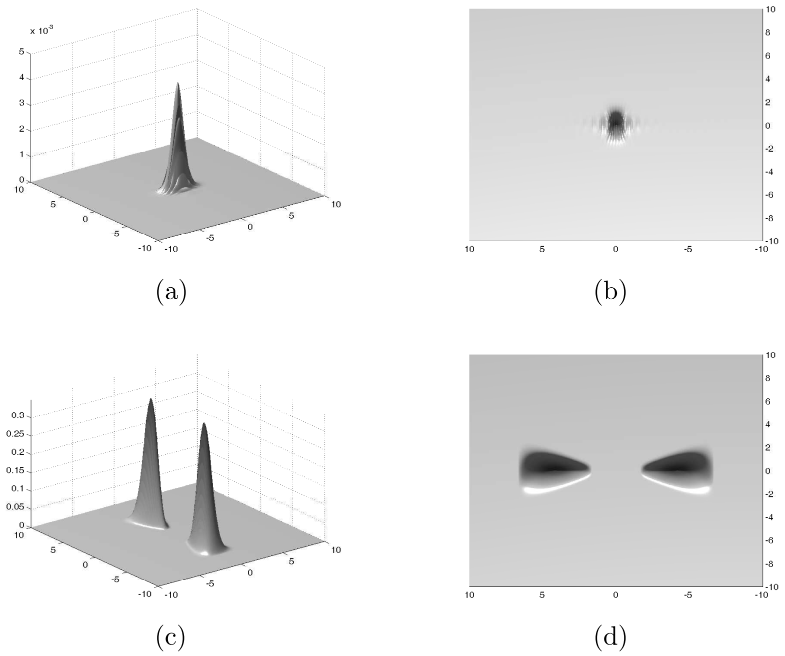

Fig. 1. This figure shows the function $\psi_{0.3,0,0}$ with ψ defined by (1), (2), and (3) in the time ((a) and (b)) and frequency ((c) and (d)) domain.

Now suppose a distribution f has a singularity in y-direction at the origin. For simplicity, we consider the linear singularity $f = \delta(x_2 - px_1)$, $p \in \mathbb{R} \setminus \{0\}$. Then the Fourier transform satisfies $\hat{f} = \delta(\xi_2 + \frac{1}{p}\xi_1)$. Notice that $\xi_2 = -\frac{1}{p}\xi_1$ is the line *perpendicular* to the line $x_2 = px_1$ along which f

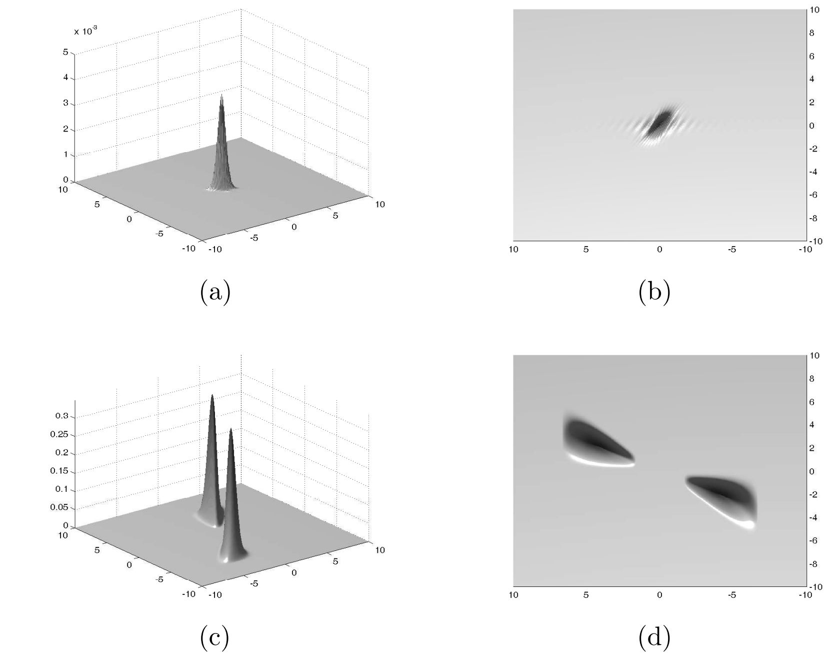

Fig. 2. This figure shows the function $\psi_{0.3,-0.5,0}$ with ψ defined by (1), (2), and (3) in the time ((a) and (b)) and frequency ((c) and (d)) domain.

is stretched. The analyzing elements of the shearlet system have "maximal overlap" with $\hat{f}$, if $\widehat{\psi_{ast}}$ is stretched along the line $\xi_2 = -\frac{1}{p}\xi_1$. But this is precisely the case if $s = -\frac{1}{p}$, which can be seen in Figures 2(c) and 2(d) for the case $p = 2$. Hence the shearlet coefficients $\langle f, \psi_{a,-0.5,0}\rangle$, i.e., those shearlet coefficients with the correct t and for which the shear parameter attains the slope of the line *perpendicular* to the direction of the singularity, would give a strong response.

We remark that we entirely focussed on the role of the shear parameter s, and thereby the role of the translation parameter t was ignored. However, the translation parameter has the same interpretation as for the Continuous Wavelet Transform, i.e., it detects the location of singularities.

§3. From Shearlets back to Wavelets

To obtain a relation between the 2-dimensional Continuous Wavelet Transform and and the Continuous Shearlet Transform, we have to slightly modify the definition of the wavelet transform by using anisotropic scaling instead isotropic scaling as follows. We define the *(anisotropic) Continuous Wavelet Transform* of some $f \in L^2(\mathbb{R}^2)$ by

$$\widetilde{\mathcal{W}}_\psi f : \mathbb{R}^+ \times \mathbb{R}^2 \to \mathbb{C}, \quad \widetilde{\mathcal{W}}_\psi f(a,t) = \langle f, T_t D_{A_a} \psi \rangle.$$

Notice that this definition is indeed a special case of the Continuous Wavelet Transform as introduced in Section 2.1.

To formulate the simple connection between shearlets and sheared anisotropic wavelets, we find it convenient to employ the notation $(g \circ S_s)(x) = g(S_s x)$ for $g \in L^2(\mathbb{R}^2)$ and $s \in \mathbb{R}$.

Lemma 1. *Let $\psi \in L^2(\mathbb{R}^2)$. Then, for all $f \in L^2(\mathbb{R}^2)$ and $(a,s,t) \in \mathbb{S}$,*

$$\mathcal{SH}_\psi f(a,s,t) = \widetilde{\mathcal{W}}_{\psi \circ S^{-1}_{s/\sqrt{a}}} f(a,t) \tag{4}$$

and

$$\mathcal{SH}_\psi f(a,s,t) = \widetilde{\mathcal{W}}_\psi (f \circ S_s)(a, S_s^{-1} t). \tag{5}$$

Proof: Let $f \in L^2(\mathbb{R}^2)$ and $(a,s,t) \in \mathbb{S}$ be given. To prove (4), we first observe that

$$A_a^{-1} S_s^{-1} = \begin{pmatrix} \frac{1}{a} & 0 \\ 0 & \frac{1}{\sqrt{a}} \end{pmatrix} \begin{pmatrix} 1 & -s \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} \frac{1}{a} & -\frac{s}{a} \\ 0 & \frac{1}{\sqrt{a}} \end{pmatrix} = S^{-1}_{\frac{s}{\sqrt{a}}} A_a^{-1}.$$

From this we can deduce that

$$\begin{aligned} \mathcal{SH}_\psi f(a,s,t) &= a^{-\frac{3}{4}} \int_{\mathbb{R}^2} f(x)\, \overline{\psi(A_a^{-1} S_s^{-1}(x-t))}\, dx \\ &= a^{-\frac{3}{4}} \int_{\mathbb{R}^2} f(x)\, \overline{\psi(S^{-1}_{s/\sqrt{a}} A_a^{-1}(x-t))}\, dx \\ &= \widetilde{\mathcal{W}}_{\psi \circ S^{-1}_{s/\sqrt{a}}} f(a,t). \end{aligned}$$

Relation (5) follows from

$$\begin{aligned} \mathcal{SH}_\psi f(a,s,t) &= a^{-\frac{3}{4}} \int_{\mathbb{R}^2} f(x)\, \overline{\psi(A_a^{-1} S_s^{-1}(x-t))}\, dx \\ &= a^{-\frac{3}{4}} \int_{\mathbb{R}^2} f(S_s x)\, \overline{\psi(A_a^{-1}(x - S_s^{-1} t))}\, dx \\ &= \widetilde{\mathcal{W}}_\psi (f \circ S_s)(a, S_s^{-1} t). \Box \end{aligned}$$

Equation (5) connects the Continuous Shearlet Transform to the 2-dimensional Continuous Wavelet Transform and gives a geometric interpretation of the Continuous Shearlet Transform and, in particular, of the action of the shear parameter: first, a shear S_s is applied to the function

f, then the *anisotropic* Continuous Wavelet Transform with scale parameter a is performed for the result and finally the transformed function is sheared back by means of S_s^{-1}. Note that the mutually inverse shear operations do not annihilate each other since the wavelet transform in between is anisotropic and thus can be understood as "enhancing" the shear in one direction. This observation can be used to implement an FFT–based Fast Continuous Shearlet Transform by using the identity

$$\widehat{\mathcal{SH}_\psi f(a,s,\cdot)}(\xi) = a^{\frac{3}{4}} \hat{f}(\xi)\, \overline{\hat{\psi}(A_a S_s^T \xi)}, \qquad (6)$$

which follows by taking the Fourier transform of (5). So once more we can make use of the FFT to accelerate the computation of a convolution–like structure which is at least as old as the FFT itself and has been used in various applications, even the fast multiplication of integers [15]; a nice introduction on how to apply the FFT in a more general framework can be found in [8].

The unavoidable technical details like discrete choice of scales and shears as well as an implementation of such a Fast Continuous Shearlet Transform will be elaborated in a forthcoming paper.

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