Frames of subspaces

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Abstract. One approach to ease the construction of frames is to first construct local components and then build a global frame from these. In this paper we will show that the study of the relation between a frame and its local components leads to the definition of a frame of subspaces. We introduce this new notion and prove that it provides us with the link we need. It will also turn out that frames of subspaces behave as a generalization of frames. In particular, we can define an analysis, a synthesis and a frame operator for a frame of subspaces, which even yield a reconstruction formula. Also concepts such as completeness, minimality, and exactness are introduced and investigated. We further study several constructions of frames of subspaces, and also of frames and Riesz frames using the theory of frames of subspaces. An important special case are harmonic frames of subspaces which generalize harmonic frames. We show that wavelet subspaces coming from multiresolution analysis belong to this class.

1. Introduction

During the last 20 years the theory of frames has been growing rapidly, since several new applications have been developed. For example, besides traditional applications as signal processing, image processing, data compression, and sampling theory, frames are now used to mitigate the effect of losses in packet-based communication systems and hence to improve the robustness of data transmission [**7, 15**], and to design high-rate constellations with full diversity in multiple-antenna code design [**17**].

To handle these emerging applications of frames new methods have to be developed. One starting point is to first build frames "locally" and then piece them together to obtain frames for the whole space. One advantage of this idea is that it would facilitate the construction of frames for special applications, since we can first construct frames or choose already known frames for smaller spaces. And in a second step one would construct a frame for the whole space from them. Therefore

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1991 *Mathematics Subject Classification.* Primary 42C15; Secondary 46C99.

Key words and phrases. Abstract frame theory, frame, harmonic frame, Hilbert space, resolution of the identity, Riesz basis, Riesz frame.

The first author was supported by NSF DMS 0102686.

The second author was supported by Forschungspreis 2003 der Universität Paderborn.

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it is necessary to derive conditions for these components, so that there exists a construction, which yields a frame for the whole space with special properties. Various approaches to piecing together familes of vectors to get a frame for the whole space have been done over the years going back to Duffin and Schaeffer's original work [**12**]. One approach used in the wavelet as well as in the Gabor case [**10, 1**] is to start with non-frame sequences and piece them together to build frames for the whole space. Another is to build frames locally and piece them together orthogonally to get frames. We refer to Heil and Walnut [**18**] for an excellent introduction to these methods and Gabor frames in general. Recently, another approach was introduced by Fornasier [**13, 14**]. Fornasier uses subspaces which are quasi-orthogonal to construct local frames and piece them together to get global frames.

In this paper we will formulate a general method for piecing together local frames to get global frames. The importance of this approach is that it is both necessary and sufficient for the the construction of global frames from local frames. Some of these results are generalizations of Fornasier's work [**13, 14**] although they were done before his papers became available to us.

Another motivation comes from the theory of C*-algebras. Just recently Casazza, Christensen, Lindner, and Vershynin [**5**] proved that the so-called "Feichtinger conjecture" is equivalent to the weak Bourgain-Tzafriri conjecture. The Feichtinger conjecture states that each bounded frame is a finite union of Riesz basic sequences. Then, Casazza and Vershynin [**8**] showed that the Kadison-Singer problem is equivalent to the strong Bourgain-Tzafriri conjecture and that these two problems have a positive solution if and only if both the Feichtinger conjecture and the F_ϵ-conjecture have positive solutions. The F_ϵ-conjecture states: For every $\epsilon > 0$, every unit norm Riesz basis is a finite union of $(1+\epsilon)$-unconditional basic sequences. A unit norm sequence $\{f_i\}_{i\in I}$ is a $(1+\epsilon)$-basic sequence if for every sequence of scalars $\{a_i\}_{i\in I}$ we have

$$(1-\epsilon)\sum_{i\in I}|a_i|^2 \le \left\|\sum_{i\in I} a_i f_i\right\|^2 \le (1+\epsilon)\sum_{i\in I}|a_i|^2.$$

To attack these problems it is important to know into which components we can divide a frame. As we will see in this paper, the necessary divisions will form a frame of subspaces for the space. At this time, it is not even known how to divide a frame into two infinite frame sequences.

In this paper we want to answer the following two questions, which relate to the two different motivations:

- Let $\{W_i\}_{i\in I}$ be a collection of closed subspaces in a Hilbert space $\mathcal{H}$ in which we want to decompose our function, where each subspace W_i is equipped with a weight v_i, which indicate its importance. When can we find frames for W_i for each $i \in I$ so that the collection of all of them is a frame with special properties for the whole space $\mathcal{H}$?
- Let $\{f_i\}_{i\in I}$ be a frame for a Hilbert space $\mathcal{H}$, and let $I = \bigcup_{j\in\mathbb{Z}} I_j$ be a partition of I so that $\{f_i\}_{i\in I_j}$ is a frame sequence for each $j \in \mathbb{Z}$. Which relations exist between the closed linear spans of $\{f_i\}_{i\in I_j}$, $j \in \mathbb{Z}$?

We start our consideration by giving a brief review of the definitions and basic properties of frames and bases and stating some notation in Section 2.

In Section 3 it will turn out that both questions above lead to the definition of a frame of subspaces. In the first subsection we will state the definition of a frame of subspaces for a given family of closed subspaces $\{W_i\}_{i\in I}$ in a Hilbert space and a family of weights $\{v_i\}_{i\in I}$. Then it is shown that this definition leads to some answers to the above questions (see Theorem 3.2), since it shows that frames of subspaces behave as a link between local components of a frame and the global structure. This will also enlighten the advantage of our approach, since now we can choose the frames for the single subspaces W_i arbitrarily and always get a frame for the whole Hilbert space by just collecting them together. Thus it differs from previous approaches and is a generalization of the approach of Fornasier [**13, 14**]. It will turn out that frames of subspaces behave as a generalization of frames. We first give a definition of completeness of a family of subspaces and show that the relation between this property and the notion of a frame of subspaces is similar to the relation between the definition of completeness of a sequence and a frame. Further in Subsection 3.2 we introduce an analysis and a synthesis operator, a frame operator, and a dual frame of subspaces for a given frame of subspaces and prove that they behave in an analogous way as the corresponding objects in abstract frame theory. We even obtain a reconstruction formula using these ingredients (Proposition 3.16). The next subsection deals with Parseval frames of subspaces, which share several properties with Parseval frames. Finally in Subsection 3.4 we show that using the theory of frames of subspaces we can construct several useful resolutions of the identity.

Section 4 deals with Riesz decompositions, which are a generalization of the notion of Riesz bases to our general setting. We further define minimality for a family of subspaces and show that it behaves as expected. Also exactness is defined in a canonical way. However, it will turn out that this property is much weaker than exactness of a frame (compare Theorem 4.6).

Some constructions are given in Section 5. Here we first state some results which help constructing frames of subspaces. An extended example concerning the situation of Gabor frames is added. In Subsection 5.2 we then show how to construct frames and Riesz frames using a frame of subspaces.

Finally, Section 6 deals with harmonic frames of subspaces. These are a generalization of harmonic frames, which distinguish themselves by having an easy construction formula. In both the finite and the infinite dimensional cases we give the definition of a harmonic frame of subspaces, state some results, and give examples, e.g., subspaces coming from Gabor systems and subspaces coming from multiresolution analysis, for their occurance.

2. Review of frames and some notation

First we will briefly recall the definitions and basic properties of frames and bases. For more information we refer to the survey articles by Casazza [**3, 4**], the books by Christensen [**9**], Gröchenig [**16**], and Young [**22**] and the research-tutorial by Heil and Walnut [**18**].

Let $\mathcal{H}$ be a separable Hilbert space and let I be an indexing set. A family $\{f_i\}_{i\in I}$ is a *frame* for $\mathcal{H}$, if there exist $0 < A \le B < \infty$ such that for all $h \in \mathcal{H}$,

$$A \, \|h\|^2 \le \sum_{i\in I} |\langle h, f_i\rangle|^2 \le B \, \|h\|^2 . \tag{2.1}$$

The constants A and B are called a *lower* and *upper frame bound* for the frame. Those sequences which satisfy only the upper inequality in (2.1) are called *Bessel sequences.* A frame is *tight*, if $A = B$. If $A = B = 1$, it is called a *Parseval frame.* We call a frame $\{f_i\}_{i\in I}$ *uniform* (or *equal norm*), if we have $\|f_i\| = \|f_j\|$ for all $i, j \in I$. A frame is *exact*, if it ceases to be a frame whenever any single element is deleted from the sequence $\{f_i\}_{i\in I}$. We say that two frames $\{f_i\}_{i\in I}$, $\{g_i\}_{i\in I}$ for $\mathcal{H}$ are *equivalent*, if there exists an invertible operator $U : \mathcal{H} \to \mathcal{H}$ satisfying $Uf_i = g_i$ for all $i \in I$. If U is a unitary operator, $\{f_i\}_{i\in I}$ and $\{g_i\}_{i\in I}$ are called *unitarily equivalent.* The *synthesis operator* $T_f : l^2(I) \to \mathcal{H}$ of a frame $f = \{f_i\}_{i\in I}$ is defined by $T_f(c) = \sum_{i\in I} c_i f_i$ for each sequence of scalars $c = \{c_i\}_{i\in I} \in l^2(I)$. The adjoint operator $T_f^* : \mathcal{H} \to l^2(I)$, the so-called *analysis operator* of $f = \{f_i\}_{i\in I}$, is given by $T_f^*(g) = \{\langle f_i, g\rangle\}_{i\in I}$. Then the *frame operator* $S_f(h) = T_f T_f^*(h) = \sum_{i\in I} \langle h, f_i\rangle f_i$ associated with $\{f_i\}_{i\in I}$ is a bounded, invertible, and positive operator mapping $\mathcal{H}$ onto itself. This provides the reconstruction formula

$$h = S_f^{-1} S_f(h) = \sum_{i\in I} \langle h, f_i\rangle \tilde{f}_i = \sum_{i\in I} \langle h, \tilde{f}_i\rangle f_i,$$

where $\tilde{f}_i = S_f^{-1} f_i$. The family $\{\tilde{f}_i\}_{i\in I}$ is also a frame for $\mathcal{H}$, called the *canonical dual frame* of $\{f_i\}_{i\in I}$. A sequence is called a *frame sequence*, if it is a frame only for its closed linear span. Moreover, we say that a frame $\{f_i\}_{i\in I}$ is a *Riesz frame*, if every subfamily of the sequence $\{f_i\}_{i\in I}$ is a frame sequence with uniform frame bounds A and B.

As important example of frames are the so-called *harmonic frames,* which are uniform Parseval frames of the form $\{U^i\varphi\}_{i\in I}$, where U is a unitary operator on $\mathcal{H}$ and $I = \{0, \ldots, N-1\}$, $N \in \mathbb{N}$ or $I = \mathbb{Z}$. Concerning a classification of harmonic frames we refer to the paper by Casazza and Kovaĉević [**7**].

Riesz bases are special cases of frames, and can be characterized as those frames which are biorthogonal to their dual frames. An equivalent definition is the following. A family $\{f_i\}_{i\in I}$ is a Riesz basis for $\mathcal{H}$, if there exist $0 < A \le B < \infty$ such that for all sequences of scalars $c = \{c_i\}_{i\in I}$,

$$A\ \|c\|_2 \le \left\| \sum_{i\in I} c_i f_i \right\| \le B\ \|c\|_2 .$$

We define the *Riesz basis constants* for $\{f_i\}_{i\in I}$ to be the largest number A and the smallest number B such that this inequality holds for all sequences of scalars c. If $\{f_i\}_{i\in I}$ is a Riesz basis only for its closed linear span, we call it a *Riesz basic sequence.*

An arbitrary sequence $\{f_i\}_{i\in I}$ in $\mathcal{H}$ is *minimal*, if $f_i \notin \overline{\text{span}}_{j\in I, j\neq i}\{f_j\}$ for all $i \in I$, or equivalently if there exists a sequence $\{\tilde{f}_i\}_{i\in I}$, which is biorthogonal to $\{f_i\}_{i\in I}$. It is *complete*, if the span of $\{f_i\}_{i\in I}$ is dense in $\mathcal{H}$.

We conclude this section by giving some notation and remarks. Throughout this paper $\mathcal{H}$ shall always denote an arbitrary separable Hilbert space. Furthermore all subspaces are assumed to be closed although this is not stated explicitely. Moreover, for the remainder a sequence $\{v_i\}_{i\in I}$ always denotes a family of weights, i.e., $v_i > 0$ for all $i \in I$.

In addition we use the following notation. Dependent on the context I denotes an indexing set or the identity operator. If W is a subspace of a Hilbert space $\mathcal{H}$, we let π_W denote the orthogonal projection of $\mathcal{H}$ onto W. If $\{e_i\}_{i\in I}$ is an orthonormal basis for $\mathcal{H}$ and $J \subset I$, π_J is the orthogonal projection of $\mathcal{H}$ onto $\overline{\text{span}}_{i\in J}\{e_i\}$.

3. Frames of subspaces

3.1. Definition and basic properties. We start with the definition of a frame of subspaces. It will turn out that frames of subspaces share many of the properties of frames, and thus can be viewed as a generalization of frames.

DEFINITION 3.1. Let I be some index set, and let $\{v_i\}_{i\in I}$ be a family of weights, i.e., $v_i > 0$ for all $i \in I$. A family of closed subspaces $\{W_i\}_{i\in I}$ of a Hilbert space $\mathcal{H}$ is a *frame of subspaces with respect to* $\{v_i\}_{i\in I}$ *for* $\mathcal{H}$, if there exist constants $0 < C \le D < \infty$ such that

$$C\|f\|^2 \le \sum_{i\in I} v_i^2 \|\pi_{W_i}(f)\|^2 \le D\|f\|^2 \quad \text{for all } f \in \mathcal{H}. \tag{3.1}$$

We call C and D the *frame bounds* for the frame of subspaces. The family $\{W_i\}_{i\in I}$ is called a C*-tight frame of subspaces with respect to* $\{v_i\}_{i\in I}$, if in (3.1) the constants C and D can be chosen so that $C = D$, a *Parseval frame of subspaces with respect to* $\{v_i\}_{i\in I}$ provided that $C = D = 1$ and an *orthonormal basis of subspaces* if $\mathcal{H} = \bigoplus_{i\in I} W_i$. Moreover, we call a frame of subspaces with respect to $\{v_i\}_{i\in I}$ v*-uniform*, if $v := v_i = v_j$ for all $i, j \in I$. If we only have the upper bound, we call $\{W_i\}_{i\in I}$ a *Bessel sequence of subspaces with respect to* $\{v_i\}_{i\in I}$ with *Bessel bound* D.

Condition (3.1) states the necessary (and also sufficient) interaction between the subspaces so that taking frames from them and putting them together yields a frame for the whole space.

The importance of this definition is that it is both necessary and sufficient for us to be able to string together frames for each of the subspaces W_i (with uniformly bounded frame constants) to get a frame for $\mathcal{H}$. This is contained in the next theorem. The implication (3) $\Rightarrow$ (1) of the following result is [**7**, Proposition 4.5]. Fornasier [**13, 14**] obtains a similar result for quasi-orthogonal decompositions.

THEOREM 3.2. *For each* $i \in I$ *let* $v_i > 0$ *and let* $\{f_{ij}\}_{j\in J_i}$ *be a frame sequence in* $\mathcal{H}$ *with frame bounds* A_i *and* B_i. *Define* $W_i = \overline{\text{span}}_{j\in J_i}\{f_{ij}\}$ *for all* $i \in I$ *and choose an orthonormal basis* $\{e_{ij}\}_{j\in J_i}$ *for each subspace* W_i. *Suppose that* $0 < A = \inf_{i\in I} A_i \le B = \sup_{i\in I} B_i < \infty$. *The following conditions are equivalent.*

(1) $\{v_i f_{ij}\}_{i\in I, j\in J_i}$ *is a frame for* $\mathcal{H}$.
(2) $\{v_i e_{ij}\}_{i\in I, j\in J_i}$ *is a frame for* $\mathcal{H}$.
(3) $\{W_i\}_{i\in I}$ *is a frame of subspaces with respect to* $\{v_i\}_{i\in I}$ *for* $\mathcal{H}$.

PROOF. Since for each $i \in I$, $\{f_{ij}\}_{j\in J_i}$ is a frame for W_i with frame bounds A_i and B_i, we obtain

$$\begin{aligned} A\sum_{i\in I} v_i^2 \|\pi_{W_i}(f)\|^2 &\le \sum_{i\in I} A_i v_i^2 \|\pi_{W_i}(f)\|^2 \le \sum_{i\in I}\sum_{j\in J_i} |\langle \pi_{W_i}(f), v_i f_{ij}\rangle|^2 \\ &\le \sum_{i\in I} B_i v_i^2 \|\pi_{W_i}(f)\|^2 \le B\sum_{i\in I} v_i^2 \|\pi_{W_i}(f)\|^2 . \end{aligned}$$

Now we observe that

$$\sum_{i\in I}\sum_{j\in J_i} |\langle \pi_{W_i}(f), v_i f_{ij}\rangle|^2 = \sum_{i\in I}\sum_{j\in J_i} |\langle f, v_i f_{ij}\rangle|^2 .$$

This shows that provided $\{v_i f_{ij}\}_{i\in I, j\in J_i}$ is a frame for $\mathcal{H}$ with frame bounds C and D, the sets $\{W_i\}_{i\in I}$ form a frame of subspaces with respect to $\{v_i\}_{i\in I}$ for $\mathcal{H}$ with frame bounds $\frac{C}{B}$ and $\frac{D}{A}$. Moreover, if $\{W_i\}_{i\in I}$ is a frame of subspaces with respect to $\{v_i\}_{i\in I}$ for $\mathcal{H}$ with frame bounds C and D, the calculation above implies that $\{v_i f_{ij}\}_{i\in I, j\in J_i}$ is a frame for $\mathcal{H}$ with frame bounds AC and BD. Thus (1) $\Leftrightarrow$ (3).

To prove the equivalence of (2) and (3), note that we can now actually calculate the orthogonal projections in the following way

$$v_i^2 \left\|\pi_{W_i}(f)\right\|^2 = v_i^2 \left\| \sum_{j\in J_i} \langle f, e_{ij}\rangle\, e_{ij} \right\|^2 = \sum_{j\in J_i} \left|\langle f, v_i e_{ij}\rangle\right|^2 .$$

From this the claim follows immediately. ☐

The definition of completeness of a sequence gives rise to a definition of completeness for a sequence of subspaces.

Definition 3.3. A family of subspaces $\{W_i\}_{i\in I}$ of $\mathcal{H}$ is called *complete*, if

$$\overline{\text{span}}_{i\in I}\{W_i\} = \mathcal{H}.$$

The next lemma possesses a well-known analog in the frame situation.

Lemma 3.4. *Let $\{W_i\}_{i\in I}$ be a family of subspaces in $\mathcal{H}$, and let $\{v_i\}_{i\in I}$ be a family of weights. If $\{W_i\}_{i\in I}$ is a frame of subspaces with respect to $\{v_i\}_{i\in I}$ for $\mathcal{H}$, then it is complete.*

Proof. Assume that $\{W_i\}_{i\in I}$ is not complete. Then there exists some $f \in \mathcal{H}$, $f \neq 0$ with $f \perp \overline{\text{span}}_{i\in I}\{W_i\}$. It follows that $\sum_{i\in I} v_i^2 \|\pi_{W_i}(f)\|^2 = 0$, hence $\{W_i\}_{i\in I}$ is not a frame of subspaces. ☐

To check completeness of a frame of subspaces, we derive the following useful characterization.

Lemma 3.5. *Let $\{W_i\}_{i\in I}$ be a family of subspaces in $\mathcal{H}$, and for each $i \in I$ let $\{e_{ij}\}_{j\in J_i}$ be an orthonormal basis for W_i. Then the following conditions are equivalent.*

(1) *$\{W_i\}_{i\in I}$ is complete.*
(2) *$\{e_{ij}\}_{i\in I, j\in J_i}$ is complete.*

Proof. The equivalence of (1) and (2) follows immediately from the definitions.

☐

If we remove an element from a frame, we obtain either another frame or an incomplete set [**9**, Theorem 5.4.7]. A similar result holds in our situation.

Proposition 3.6. *The removal of a subspace from a frame of subspaces with respect to some family of weights leaves either a frame of subspaces with respect to the same family of weights or an incomplete family of subspaces.*

Proof. Let $\{W_i\}_{i\in I}$ be a frame of subspaces with respect to $\{v_i\}_{i\in I}$ for $\mathcal{H}$, and for each $i \in I$ let $\{e_{ij}\}_{j\in J_i}$ be an orthonormal basis for W_i. By Theorem 3.2, $\{v_i e_{ij}\}_{i\in I, j\in J_i}$ is a frame for $\mathcal{H}$. Let $i_0 \in I$. By [**9**, Theorem 5.4.7], $\{v_i e_{ij}\}_{i\in I\setminus\{i_0\}, j\in J_i}$ is either a frame or an incomplete set. If it is a frame, again by Theorem 3.2, also $\{W_i\}_{i\in I\setminus\{i_0\}}$ is a frame of subspaces with respect to $\{v_i\}_{i\in I}$ for $\mathcal{H}$.

Now suppose that $\{v_i e_{ij}\}_{i\in I\setminus\{i_0\}, j\in J_i}$ and hence $\{e_{ij}\}_{i\in I\setminus\{i_0\}, j\in J_i}$ is an incomplete set. By Lemma 3.5, also $\{W_i\}_{i\in I\setminus\{i_0\}}$ is incomplete. $\square$

We further observe that the intersection of the elements of a frame of subspaces with a subspace still leaves a frame of subspaces for a smaller space.

LEMMA 3.7. *Let V be a subspace of $\mathcal{H}$ and let $\{W_i\}_{i\in I}$ be a frame of subspaces with respect to $\{v_i\}_{i\in I}$ for $\mathcal{H}$ with frame bounds C and D. Then $\{W_i\cap V\}_{i\in I}$ is a frame of subspaces with respect to $\{v_i\}_{i\in I}$ for V with frame bounds C and D.*

PROOF. For all $f\in V$ we have
$$\sum_{i\in I} v_i^2\|\pi_{W_i}(f)\|^2 = \sum_{i\in I} v_i^2\|\pi_{W_i\cap V}(f)\|^2.$$
From this the result follows at once. $\square$

3.2. Frame properties. In this subsection we will show that a frame of subspaces behaves as a generalization of a frame, thus providing an associated analysis and synthesis operator, a frame operator and a dual object.

For the definition of an analysis and a synthesis operator for a frame of subspaces, we will need the following notation.

NOTATION 3.8. For each family of subspaces $\{W_i\}_{i\in I}$ of $\mathcal{H}$, we define the space $\left(\sum_{i\in I}\oplus W_i\right)_{\ell_2}$ by
$$\left(\sum_{i\in I}\oplus W_i\right)_{\ell_2} = \{\{f_i\}_{i\in I} | f_i\in W_i \text{ and } \sum_{i\in I}\|f_i\|^2<\infty\}$$
with inner product given by
$$\langle\{f_i\}_{i\in I},\{g_i\}_{i\in I}\rangle = \sum_{i\in I}\langle f_i, g_i\rangle.$$

We start with the definition of a synthesis operator for a frame of subspaces. To show that the series appearing in this formula converges unconditionally, we need the next lemma.

LEMMA 3.9. *Let $\{W_i\}_{i\in I}$ be a Bessel sequence of subspaces with respect to $\{v_i\}_{i\in I}$ for $\mathcal{H}$. Then, for each sequence $\{f_i\}_{i\in I}$ with $f_i\in W_i$ for each $i\in I$, the series $\sum_{i\in I} v_i f_i$ converges unconditionally.*

PROOF. Let $f=\{f_i\}_{i\in I}\in\left(\sum_{i\in I}\oplus W_i\right)_{\ell_2}$. Fix $J\subset I$ with $|J|<\infty$ and let $g=\sum_{i\in J} v_i f_i$. Then we compute
$$\|\sum_{i\in J} v_i f_i\|^4 = \left(\langle g, \sum_{i\in J} v_i f_i\rangle\right)^2 = \left(\sum_{i\in J} v_i\langle\pi_{W_i}(g), f_i\rangle\right)^2 \le \left(\sum_{i\in J} v_i\|\pi_{W_i}(g)\|\|f_i\|\right)^2$$
$$\le \sum_{i\in J} v_i^2\|\pi_{W_i}(g)\|^2\sum_{i\in J}\|f_i\|^2 \le D\|g\|^2\sum_{i\in J}\|f_i\|^2 \le D\|\sum_{i\in J} v_i f_i\|^2\|f\|^2.$$
Hence,
$$\|\sum_{i\in J} v_i f_i\|^2 \le D\|f\|^2.$$
It follows that $\sum_{i\in I} v_i f_i$ is weakly unconditionally Cauchy and hence unconditionally convergent in $\mathcal{H}$ (see [**11**], page 44, Theorems 6 and 8). $\square$

DEFINITION 3.10. Let $\{W_i\}_{i\in I}$ be a frame of subspaces with respect to $\{v_i\}_{i\in I}$ for $\mathcal{H}$. Then the *synthesis operator* for $\{W_i\}_{i\in I}$ and $\{v_i\}_{i\in I}$ is the operator

$$T_{W,v} : \left(\sum_{i\in I} \oplus W_i\right)_{\ell_2} \longrightarrow \mathcal{H}$$

defined by

$$T_{W,v}(f) = \sum_{i\in I} v_i f_i \quad \text{for all } f = \{f_i\}_{i\in I} \in (\sum_{i\in I} \oplus W_i)_{\ell_2}.$$

We call the adjoint $T^*_{W,v}$ of the synthesis operator the *analysis operator*.

The following proposition will provide us with a concrete formula for the analysis operator.

PROPOSITION 3.11. *Let $\{W_i\}_{i\in I}$ be a frame of subspaces with respect to $\{v_i\}_{i\in I}$ for $\mathcal{H}$. Then the analysis operator $T^*_{W,v} : \mathcal{H} \to (\sum_{i\in I} \oplus W_i)_{\ell_2}$ is given by*

$$T^*_{W,v}(f) = \{v_i \pi_{W_i}(f)\}_{i\in I}.$$

PROOF. Let $f \in \mathcal{H}$ and $g = \{g_i\}_{i\in I} \in (\sum_{i\in I} \oplus W_i)_{\ell_2}$. Using the definition of $T_{W,v}$ we compute that

$$\langle T^*_{W,v}(f), g\rangle = \langle f, T_{W,v}(g)\rangle = \langle f, \sum_{i\in I} v_i g_i\rangle = \sum_{i\in I} v_i \langle f, g_i\rangle.$$

Since $g_i \in W_i$ for each $i \in I$, we can continue in the following way:

$$\sum_{i\in I} v_i \langle f, g_i\rangle = \sum_{i\in I} v_i \langle \pi_{W_i}(f), g_i\rangle = \langle \{v_i \pi_{W_i}(f)\}_{i\in I}, \{g_i\}_{i\in I}\rangle.$$

□

The well-known relations between a frame and the associated analysis and synthesis operator also holds in our more general situation.

THEOREM 3.12. *Let $\{W_i\}_{i\in I}$ be a family of subspaces in $\mathcal{H}$, and let $\{v_i\}_{i\in I}$ be a family of weights. Then the following conditions are equivalent.*

(1) *$\{W_i\}_{i\in I}$ is a frame of subspaces with respect to $\{v_i\}_{i\in I}$ for $\mathcal{H}$.*
(2) *The synthesis operator $T_{W,v}$ is bounded, linear and onto.*
(3) *The analysis operator $T^*_{W,v}$ is a (possibly into) isomorphism.*

PROOF. First we prove (1) $\Leftrightarrow$ (3). This claim follows immediately from the fact that for each $f \in \mathcal{H}$ we have

$$\|T^*_{W,v}(f)\|^2 = \|\{v_i \pi_{W_i}(f)\}_{i\in I}\|^2 = \sum_{i\in I} v_i^2 \|\pi_{W_i}(f)\|^2.$$

Further recall that (2) $\Leftrightarrow$ (3) holds in general for each operator on a Hilbert space. □

In an analogous way as in frame theory we can define equivalence classes of frames of subspaces. Using the synthesis operator we can also characterize exactly the elements belonging to the same equivalence class.

DEFINITION 3.13. Let $\{W_i\}_{i\in I}$ and $\{\widetilde{W}_i\}_{i\in I}$ be frames of subspaces with respect to the same family of weights. We say that they are *(unitarily) equivalent*, if there exists an (unitary) invertible operator U on $\mathcal{H}$ such that $W_i = U(\widetilde{W}_i)$ for all $i \in I$.

LEMMA 3.14. *Let $\{W_i\}_{i\in I}$ and $\{\widetilde{W}_i\}_{i\in I}$ be frames of subspaces with respect to the same family of weights $\{v_i\}_{i\in I}$. The following conditions are equivalent.*

(1) *$\{W_i\}_{i\in I}$ and $\{\widetilde{W}_i\}_{i\in I}$ are (unitarily) equivalent.*
(2) *There exists an (unitary) invertible operator U on $\mathcal{H}$ such that $T_{W,v} = U^{-1}T_{\widetilde{W},v}U$, where U is applied to each component.*

PROOF. This follows immediately from the definition of the synthesis operator. □

As in the well-known frame situation, there also exists an associated frame operator for each frame of subspaces which satisfies similar properties as we will see in the next proposition. For instance we even obtain a reconstruction formula.

DEFINITION 3.15. Let $\{W_i\}_{i\in I}$ be a frame of subspaces with respect to $\{v_i\}_{i\in I}$ for $\mathcal{H}$. Then the frame operator $S_{W,v}$ for $\{W_i\}_{i\in I}$ and $\{v_i\}_{i\in I}$ is defined by

$$S_{W,v}(f) = T_{W,v}T^*_{W,v}(f) = T_{W,v}(\{v_i\pi_{W_i}(f)\}_{i\in I}) = \sum_{i\in I} v_i^2 \pi_{W_i}(f).$$

The next proposition generalizes a result of Fornasier [**13, 14**].

PROPOSITION 3.16. *Let $\{W_i\}_{i\in I}$ be a frame of subspaces with respect to $\{v_i\}_{i\in I}$ with frame bounds C and D. Then the frame operator $S_{W,v}$ for $\{W_i\}_{i\in I}$ and $\{v_i\}_{i\in I}$ is a positive, self-adjoint, invertible operator on $\mathcal{H}$ with $CI \le S_{W,v} \le DI$. Further, we have the reconstruction formula*

$$f = T_{S^{-1}_{W,v}W,v}T^*_{W,v}(f) = \sum_{i\in I} v_i^2 S^{-1}_{W,v}\pi_{W_i}(f) \quad \textit{for all } f \in \mathcal{H}.$$

PROOF. For any $f \in \mathcal{H}$, we have

$$\langle S_{W,v}(f), f\rangle = \langle \sum_{i\in I} v_i^2 \pi_{W_i}(f), f\rangle = \sum_{i\in I} v_i^2 \langle \pi_{W_i}(f), f\rangle = \sum_{i\in I} v_i^2 \|\pi_{W_i}(f)\|^2,$$

which implies that $S_{W,v}$ is a positive operator. We further compute

$$\langle Cf, f\rangle = C\|f\|^2 \le \sum_{i\in I} v_i^2 \|\pi_{W_i}(f)\|^2 = \langle S_{W,v}(f), f\rangle \le \langle Df, f\rangle.$$

This shows that $CI \le S_{W,v} \le DI$ and hence $S_{W,v}$ is an invertible operator on $\mathcal{H}$. Furthermore, for any $f, g \in \mathcal{H}$ we have

$$\langle S_{W,v}(f), g\rangle = \sum_{i\in I} v_i^2 \langle \pi_{W_i}(f), g\rangle = \sum_{i\in I} v_i^2 \langle f, \pi_{W_i}(g)\rangle.$$

Thus $S_{W,v}$ is self-adjoint. At last the reconstruction formula follows immediately from

$$f = S^{-1}_{W,v}S_{W,v}(f) = \sum_{i\in I} v_i^2 S^{-1}_{W,v}\pi_{W_i}(f).$$

□

The following result will show the connection between the frame operator for a frame of subspaces and the frame operator for the frame generated by orthonormal bases of the subspaces. Also the connection between the reconstruction formulas is exposed.

PROPOSITION 3.17. *Let $\{W_i\}_{i\in I}$ be a frame of subspaces with respect to $\{v_i\}_{i\in I}$ for $\mathcal{H}$ and $\{v_i f_{ij}\}_{j\in J_i}$ be a Parseval frame for W_i for each $i \in I$. Then the frame operator $S_{W,v}$ equals the frame operator S_{vf} for the frame $\{v_i f_{ij}\}_{i\in I, j\in J_i}$, and for all $g \in \mathcal{H}$ we have*

$$\sum_{i\in I} v_i^2 S_{W,v}^{-1} \pi_{W_i}(g) = \sum_{i\in I, j\in J_i} \langle g, v_i f_{ij}\rangle \, S_{vf}^{-1} v_i f_{ij}.$$

PROOF. Since $\{f_{ij}\}_{j\in J_i}$ is a Parseval frame for W_i for all $i \in I$, if $g \in \mathcal{H}$ then

$$\pi_{W_i}(g) = \sum_{j\in J_i} \langle \pi_{W_i}(g), f_{ij}\rangle f_{ij} = \sum_{j\in J_i} \langle g, f_{ij}\rangle f_{ij}.$$

Thus

$$S_{W,v}(g) = \sum_{i\in I} v_i^2 \pi_{W_i}(g) = \sum_{i\in I, j\in J_i} \langle g, v_i f_{ij}\rangle v_i f_{ij} = S_{vf}(g).$$

Moreover, we obtain

$$\sum_{i\in I} v_i^2 S_{W,v}^{-1} \pi_{W_i}(g) = \sum_{i\in I} S_{vf}^{-1} \sum_{j\in J_i} \langle g, v_i f_{ij}\rangle \, v_i f_{ij} = \sum_{i\in I, j\in J_i} \langle g, v_i f_{ij}\rangle \, S_{vf}^{-1} v_i f_{ij}.$$

□

Using the frame operator for a frame of subspaces for a special subspace yields an easy way to compute the orthogonal projection onto this subspace.

PROPOSITION 3.18. *Let $\{W_i\}_{i\in I}$ be a frame of subspaces with respect to $\{v_i\}_{i\in I}$ for a subspace V of $\mathcal{H}$. Then, the orthogonal projection π_V onto V is given by*

$$\pi_V(f) = \sum_{i\in I} v_i^2 S_{W,v}^{-1} \pi_{W_i}(f) \quad \textit{for all } f \in \mathcal{H}.$$

PROOF. The fact that $S_{W,v} : V \to V$ implies that $\pi_V(f) = 0$ for all $f \in V^{\perp}$. By Proposition 3.16, we have

$$f = \sum_{i\in I} v_i^2 S_{W,v}^{-1} \pi_{W_i}(f) \quad \text{for all } f \in V.$$

Thus $\pi_V^2 = \pi_V$, which finishes the proof. □

In the same manner as in frame theory we define a dual frame of subspaces associated with a frame of subspaces.

DEFINITION 3.19. Let $\{W_i\}_{i\in I}$ be a frame of subspaces with respect to $\{v_i\}_{i\in I}$ and with frame operator $S_{W,v}$. Then $\{S_{W,v}^{-1} W_i\}_{i\in I}$ is called the *dual frame of subspaces with respect to* $\{v_i\}_{i\in I}$.

The dual frame of subspaces is a frame of subspaces with the same weights. In fact, more is true.

PROPOSITION 3.20. *Let $\{W_i\}_{i\in I}$ be a frame of subspaces with respect to $\{v_i\}_{i\in I}$, and let $T : \mathcal{H} \to \mathcal{H}$ be an invertible operator on $\mathcal{H}$. Then $\{TW_i\}_{i\in I}$ is a frame of subspaces with respect to $\{v_i\}_{i\in I}$.*

PROOF. Since T is an invertible operator on $\mathcal{H}$, we have that $\pi_{TW_i} = T\pi_{W_i}T^{-1}$. Let $C, D > 0$ be the frame bounds for the frame of subspaces $\{W_i\}_{i\in I}$. Then for all $f \in \mathcal{H}$ we have

$$\sum_{i\in I} v_i^2 \|\pi_{TW_i}(f)\|^2 = \sum_{i\in I} v_i^2 \|T\pi_{W_i}T^{-1}(f)\|^2 \le \|T\|^2 \sum_{i\in I} v_i^2 \|\pi_{W_i}T^{-1}(f)\|^2$$
$$\le \|T\|^2 D \|T^{-1}(f)\|^2 \le \|T\|^2 \|T^{-1}\|^2 D \|f\|^2.$$

Similarly, we obtain a lower frame of subspaces bound for $\{TW_i\}_{i\in I}$. □

3.3. Parseval frames of subspaces. Parseval frames play an important role in abstract frame theory, since they are extremely useful for applications. Therefore in this subsection we study characterizations of Parseval frames of subspaces and special cases of them.

The first result extends [**7**, Corollary 4.1].

COROLLARY 3.21. *For each $i \in I$ let $v_i > 0$ and let $\{f_{ij}\}_{j\in J_i}$ be a Parseval frame sequence in $\mathcal{H}$. Define $W_i = \overline{\text{span}}_{j\in J_i}\{f_{ij}\}$ for all $i \in I$, and choose for each subspace W_i an orthonormal basis $\{e_{ij}\}_{j\in J_i}$. Then the following conditions are equivalent.*

(1) *$\{v_i f_{ij}\}_{i\in I, j\in J_i}$ is a Parseval frame for $\mathcal{H}$.*
(2) *$\{v_i e_{ij}\}_{i\in I, j\in J_i}$ is a Parseval frame for $\mathcal{H}$.*
(3) *$\{W_i\}_{i\in I}$ is a Parseval frame of subspaces with respect to $\{v_i\}_{i\in I}$ for $\mathcal{H}$.*

PROOF. This follows immediately from Theorem 3.2. □

We can also characterize Parseval frames of subspaces in terms of their frame operators in a similar manner as in frame theory.

PROPOSITION 3.22. *Let $\{W_i\}_{i\in I}$ be a family of subspaces in $\mathcal{H}$, and let $\{v_i\}_{i\in I}$ be a family of weights. Then the following conditions are equivalent.*

(1) *$\{W_i\}_{i\in I}$ is a Parseval frame of subspaces with respect to $\{v_i\}_{i\in I}$ for $\mathcal{H}$.*
(2) *$S_{W,v} = I$.*

PROOF. For each $i \in I$, let $\{e_{ij}\}_{j\in J_i}$ be an orthonormal basis for W_i.

By Proposition 3.16, (1) implies (2). To prove the converse implication suppose that $S_{W,v} = I$. Then for all $f \in \mathcal{H}$ we have

$$f = S_{W,v}(f) = \sum_{i\in I} v_i^2 \pi_{W_i}(f) = \sum_{i\in I} v_i^2 \sum_{j\in J_i} \langle f, e_{ij}\rangle e_{ij}.$$

This yields

$$\|f\|^2 = \left\langle \sum_{i\in I} v_i^2 \sum_{j\in J_i} \langle f, e_{ij}\rangle e_{ij}, f \right\rangle = \sum_{i\in I} v_i^2 \|\pi_{W_i}(f)\|^2.$$

□

We also have the following characterization of orthonormal bases of subspaces, which reflects exactly the situation in frame theory.

PROPOSITION 3.23. *Let $\{W_i\}_{i\in I}$ be a family of subspaces in $\mathcal{H}$, and let $\{v_i\}_{i\in I}$ be a family of weights. Then the following conditions are equivalent.*

(1) *$\{W_i\}_{i\in I}$ is an orthonormal basis of subspaces for $\mathcal{H}$.*
(2) *$\{W_i\}_{i\in I}$ is a 1-uniform Parseval frame of subspaces for $\mathcal{H}$.*

PROOF. For each $i \in I$, let $\{e_{ij}\}_{j\in J_i}$ be an orthonormal basis for W_i.

If (1) is satisfied, then $\{e_{ij}\}_{i\in I, j\in J_i}$ is an orthonormal basis for $\mathcal{H}$. This implies

$$\|f\|^2 = \sum_{i\in I}\sum_{j\in J_i} |\langle e_{ij}, f\rangle|^2 = \sum_{i\in I} \|\pi_{W_i}(f)\|^2$$

for all $f \in \mathcal{H}$. Thus also (2) holds.

On the other hand suppose that (2) holds. Then for all $f \in \mathcal{H}$ we have

$$\|f\|^2 = \sum_{i\in I} \|\pi_{W_i}(f)\|^2 = \sum_{i\in I}\sum_{j\in J_i} |\langle e_{ij}, f\rangle|^2$$

and $\|e_{ij}\| = 1$ for all $i \in I, j \in J_i$, which shows that $\{e_{ij}\}_{i\in I, j\in J_i}$ is an orthonormal basis for $\mathcal{H}$. This immediately implies $\mathcal{H} = \bigoplus_{i\in I} W_i$, hence (1) follows. □

3.4. Resolution of the identity. Let $\{W_i\}_{i\in I}$ be a frame of subspaces with respect to $\{v_i\}_{i\in I}$ for $\mathcal{H}$ and let its frame operator be denoted by $S_{W,v}$. By Proposition 3.16, we have

$$f = \sum_{i\in I} v_i^2 S_{W,v}^{-1} \pi_{W_i}(f) \quad \text{for all } f \in \mathcal{H}.$$

This shows that the family of operators $\{v_i^2 S_{W,v}^{-1} \pi_{W_i}\}_{i\in I}$ is a resolution of the identity. But a frame of subspaces for $\mathcal{H}$ provides us with many more resolutions of the identity than only this one.

We start our consideration with the general definition of a resolution of the identity.

DEFINITION 3.24. Let I be an indexing set. A family of bounded operators $\{T_i\}_{i\in I}$ on $\mathcal{H}$ is called a *(unconditional) resolution of the identity* on $\mathcal{H}$, if for all $f \in \mathcal{H}$ we have

$$f = \sum_{i\in I} T_i(f)$$

(and the series converges unconditionally for all $f \in \mathcal{H}$).

Note that it follows from the definition and the uniform boundedness principle that $\sup_{i\in I} \|T_i\| < \infty$.

The following result shows another way to obtain a resolution of the identity from a frame of subspaces, which even satisfies an analog of (3.1).

PROPOSITION 3.25. *Let $\{v_i\}_{i\in I}$ be a family of weights, and for each $i \in I$ let $\{v_i f_{ij}\}_{j\in J_i}$ be a frame sequence in $\mathcal{H}$ with frame bounds A_i and B_i. Suppose that $\{W_i\}_{i\in I}$ is a frame of subspaces with respect to $\{v_i\}_{i\in I}$ for $\mathcal{H}$ with frame bounds C and D, where $W_i = \overline{\operatorname{span}}_{j\in J_i}\{f_{ij}\}$ for all $i \in I$. Then $\{v_i f_{ij}\}_{i\in I, j\in J_i}$ is a frame for $\mathcal{H}$ with frame operator denoted by S_{vf}. Further, for each $i \in I$, let $T_i : \mathcal{H} \to W_i$ be given by*

$$T_i(f) = \sum_{j\in J_i} \langle f, S_{vf}^{-1} v_i f_{ij}\rangle v_i f_{ij}.$$

If $0 < A = \inf_{i\in I} A_i \le B = \sup_{i\in I} B_i < \infty$, then $\{T_i\}_{i\in I}$ is an unconditional resolution of the identity on $\mathcal{H}$ satisfying

$$\frac{AC}{B^2 D^2}\|f\|^2 \le \sum_{i\in I} v_i^2 \|T_i(f)\|^2 \le \frac{B^2 D^3}{A^2 C^2}\|f\|^2 \quad \textit{for all } f \in \mathcal{H}.$$

PROOF. Recall that $\{v_i f_{ij}\}_{i\in I, j\in J_i}$ is a frame for $\mathcal{H}$ by Theorem 3.2 with frame bounds AC and BD. For any $f \in \mathcal{H}$ we have

$$f = \sum_{i\in I}\sum_{j\in J_i} \langle f, S_{vf}^{-1} v_i f_{ij}\rangle v_i f_{ij} = \sum_{i\in I} T_i(f).$$

Since this is convergence relative to a frame, the convergence is unconditional.

For each $i \in I$, let $S_{vf,i}$ be the frame operator for $\{v_i f_{ij}\}_{j\in J_i}$. Let $i \in I$ be fixed. Then we obtain

$$\begin{aligned} \|T_i(f)\|^2 &= \|\sum_{j\in J_i} \langle S_{vf}^{-1} f, v_i f_{ij}\rangle v_i f_{ij}\|^2 \\ &= \|S_{vf,i}\pi_{W_i} S_{vf}^{-1}(f)\|^2 \\ &\le \|S_{vf,i}\|^2 \|\pi_{W_i} S_{vf}^{-1}(f)\|^2. \end{aligned}$$

To prove the upper bound, we compute

$$\sum_{i\in I} v_i^2 \|T_i(f)\|^2 \le B^2D^2 \sum_{i\in I} v_i^2 \|\pi_{W_i} S_{vf}^{-1}(f)\|^2 \le B^2 D^3 \|S_{vf}^{-1}(f)\|^2 \le \frac{B^2D^3}{A^2C^2}\|f\|^2.$$

The lower bound follows from

$$\begin{aligned} \sum_{i\in I} v_i^2 \|T_i(f)\|^2 &= \sum_{i\in I} v_i^2 \|S_{vf,i}\pi_{W_i} S^{-1}(f)\|^2 \ge \sum_{i\in I} A_i v_i^2 \|\pi_{W_i} S_{vf}^{-1}(f)\|^2 \\ &\ge AC\|S_{vf}^{-1}(f)\|^2 \ge \frac{AC}{B^2D^2}\|f\|^2. \end{aligned}$$

□

We now give another method for obtaining an unconditional resolution of the identity from a frame of subspaces. A special case of this can be found in Fornasier [**13, 14**].

PROPOSITION 3.26. *Let $\{W_i\}_{i\in I}$ be a frame of subspaces with respect to $\{v_i\}_{i\in I}$ for $\mathcal{H}$ with frame bounds C and D, and let $S_{W,v}$ denote its frame operator. Then $\{T_i\}_{i\in I}$ defined by $T_i = \pi_{W_i} S_{W,v}^{-1}$, $i \in I$ satisfies that $\{v_i^2 T_i\}_{i\in I}$ is an unconditional resolution of the identity, and for all $f \in \mathcal{H}$ we have*

$$\frac{C}{D^2}\|f\|^2 \le \sum_{i\in I} v_i^2 \|T_i(f)\|^2 \le \frac{D}{C^2}\|f\|^2.$$

PROOF. First, for any $f \in \mathcal{H}$ we have

$$\sum_{i\in I} v_i^2 \pi_{W_i} S_{W,v}^{-1}(f) = S_{W,v} S_{W,v}^{-1}(f) = f.$$

To prove the second claim we compute

$$\frac{C}{D^2}\|f\|^2 \le C\|S_{W,v}^{-1}(f)\|^2 \le \sum_{i\in I} v_i^2 \|\pi_{W_i} S_{W,v}^{-1}(f)\|^2 \le D\|S_{W,v}^{-1}(f)\|^2 \le \frac{D}{C^2}\|f\|^2.$$

□

The next result will turn out to be useful for proving a lower bound for $\sum_{i\in I} v_i^2 \|T_i(f)\|^2$ if $\{v_i^2 T_i\}_{i\in I}$ is a resolution of the identity.

LEMMA 3.27. *Let $\{W_i\}_{i\in I}$ be a frame of subspaces with respect to $\{v_i\}_{i\in I}$ with frame bounds C and D for $\mathcal{H}$, and let $T_i : \mathcal{H} \to W_i$ be such that $\{v_i^2 T_i\}_{i\in I}$ is a resolution of the identity on $\mathcal{H}$ (Note that a resolution of the identity need not be unconditional so the index set must have an ordering on it. In our case, the result will hold for any ordering so we do not specify the ordering here). For any $J \subset I$ we have*

$$\frac{1}{D}\|\sum_{j\in J} v_i^2 T_j(f)\|^2 \le \sum_{j\in J} v_j^2 \|T_j(f)\|^2 \quad \textit{for all } f \in \mathcal{H}.$$

PROOF. We may assume that $|J| < \infty$, since if our inequality holds for all finite subsets then it holds for all subsets. Let $f \in \mathcal{H}$ and set $g = \sum_{j\in J} v_j^2 T_j(f)$. Then, using the fact that $\{W_i\}_{i\in I}$ is a frame of subspaces with respect to $\{v_i\}_{i\in I}$ for $\mathcal{H}$, we compute

$$\begin{aligned}
\|g\|^4 &= \left(\langle g, \sum_{j\in J} v_j^2 T_j(f)\rangle\right)^2 \\
&= \left(\sum_{j\in J} v_j \langle g, v_j T_j(f)\rangle\right)^2 \\
&= \left(\sum_{j\in J} v_j \langle \pi_{W_j}(g), v_j T_j(f)\rangle\right)^2 \\
&\le \left(\sum_{j\in J} v_j \|\pi_{W_j}(g)\| \|v_j T_j(f)\|\right)^2 \\
&\le \sum_{j\in J} v_j^2 \|\pi_{W_j}(g)\|^2 \sum_{j\in J} \|v_j T_j(f)\|^2 \\
&\le D\|g\|^2 \sum_{j\in J} \|v_j T_j(f)\|^2.
\end{aligned}$$

Dividing both sides of this inequality by $D\|g\|^2$ completes the proof.

□

Using this lemma, we obtain bounds for $\sum_{i\in I} v_i^2 \|T_i(f)\|^2$ for many resolutions of the identity $\{v_i^2 T_i\}_{i\in I}$.

PROPOSITION 3.28. *Let $\{W_i\}_{i\in I}$ be a frame of subspaces with respect to $\{v_i\}_{i\in I}$ for $\mathcal{H}$ with frame bounds C and D, let $T_i : \mathcal{H} \to W_i$ be such that $\{v_i^2 T_i\}_{i\in I}$ is a resolution of the identity on $\mathcal{H}$, and assume that $T_i \pi_{W_i} = T_i$. Then for all $f \in \mathcal{H}$*

$$\frac{1}{D}\|f\|^2 \le \sum_{i\in I} v_i^2 \|T_i(f)\|^2 \le DE\|f\|^2,$$

where $E = sup_i \|T_i\| < \infty$.

PROOF. By Lemma 3.27, for all $f \in \mathcal{H}$, we have

$$\frac{1}{D}\|f\|^2 = \frac{1}{D}\|\sum_{i\in I} v_i^2 T_i(f)\|^2 \le \sum_{i\in I} v_i^2 \|T_i(f)\|^2 = \sum_{i\in I} v_i^2 \|T_i \pi_{W_i}(f)\|^2$$

$$\leq \sum_{i\in I} v_i^2 \|T_i\|^2 \|\pi_{W_i}(f)\|^2 \leq E \sum_{i\in I} v_i^2 \|\pi_{W_i}(f)\|^2 \leq DE\|f\|^2.$$

□

Obviously the condition $T_i\pi_{W_i} = T_i$ for all $i \in I$ is satisfied by the example $\{v_i^2 S_{W,v}^{-1}\pi_{W_i}\}_{i\in I}$ from the beginning of this subsection. This shows that this family of operators is not only a resolution of the identity but even satisfies an analog of (3.1).

The following definition provides us with a condition, which implies that a resolution of the identity $T_i : \mathcal{H} \to W_i$, where $\{W_i\}_{i\in I}$ is a frame of subspaces with respect to $\{v_i\}_{i\in I}$ for $\mathcal{H}$, automatically satisfies an analog of (3.1).

DEFINITION 3.29. A family of bounded operators $\{T_i\}_{i\in I}$ on $\mathcal{H}$ is called an *ℓ^2-resolution of the identity with respect to a family of weights* $\{v_i\}_{i\in I}$ on $\mathcal{H}$, if it is a resolution of the identity on $\mathcal{H}$ and there exists a constant $B > 0$ so that for all $f \in \mathcal{H}$ we have

$$\sum_{i\in I} v_i^{-2} \|T_i(f)\|^2 \leq B\|f\|^2.$$

THEOREM 3.30. *Let $\{W_i\}_{i\in I}$ be a frame of subspaces with respect to $\{v_i\}_{i\in I}$ for $\mathcal{H}$, and let $T_i : \mathcal{H} \to W_i$ be such that $\{v_i^2 T_i\}_{i\in I}$ is an ℓ^2-resolution of the identity with respect to $\{v_i\}_{i\in I}$ on $\mathcal{H}$. Then there exist constants $A, B > 0$ so that for all $f \in \mathcal{H}$ we have*

$$A\|f\|^2 \leq \sum_{i\in I} v_i^2 \|T_i(f)\|^2 \leq B\|f\|^2.$$

PROOF. This follows immediately from the definition of an ℓ^2-resolution of the identity on $\mathcal{H}$ and Lemma 3.27. □

4. Riesz decompositions

In this section we first study minimal frames of subspaces, which share similar properties with minimal frames.

DEFINITION 4.1. A family of subspaces $\{W_i\}_{i\in I}$ of $\mathcal{H}$ is called *minimal*, if for each $i \in I$

$$W_i \cap \overline{\operatorname{span}}_{j\in I, j\neq i}\{W_j\} = \{0\}.$$

Using orthonormal bases for the subspaces we obtain a useful characterization of minimal families of subspaces.

LEMMA 4.2. *Let $\{W_i\}_{i\in I}$ be a family of subspaces in $\mathcal{H}$, and for each $i \in I$ let $\{e_{ij}\}_{j\in J_i}$ be an orthonormal basis for W_i. Then the following conditions are equivalent.*

(1) *$\{W_i\}_{i\in I}$ is minimal.*
(2) *$\{e_{ij}\}_{i\in I, j\in J_i}$ is minimal.*

PROOF. The implication (2) ⇒ (1) is obvious. To prove (1) ⇒ (2) suppose that $\{c_{ij}\}_{j\in J_i} \in \ell_2(J_i)$ for all $i \in I$ and we have $f_i = \sum_{j\in J_i} c_{ij} e_{ij}$ and $f = \{f_i\}_{i\in I} \in \left(\sum_{i\in I} \oplus W_i\right)_{\ell_2}$. If $\sum_{i\in I} f_i = 0$, then by minimality of $\{W_i\}_{i\in I}$ we have that $f_i = 0$ for all $i \in I$ and so $c_{ij} = 0$ for all $i \in I, j \in J_i$. It follows that $\{e_{ij}\}_{i\in I, j\in J_i}$ is a minimal frame for $\mathcal{H}$. □

The following two propositions show that for families of subspaces we can also give a definition of biorthogonal families of subspaces, which possess similar properties compared to minimal frames of subspaces as in the situation of minimal frames (compare [**9**, Lemma 3.3.1]).

PROPOSITION 4.3. *Let $\{W_i\}_{i\in I}$ be a family of subspaces in $\mathcal{H}$. Then the following conditions are equivalent.*

(1) *$\{W_i\}_{i\in I}$ is minimal.*
(2) *There exists a unique maximal (up to containment) biorthogonal family of subspaces for $\{W_i\}_{i\in I}$, i.e., there exists a family of subspaces $\{V_i\}_{i\in I}$ with $W_i \perp V_j$ for all $i,j \in I$, $j \neq i$ and $f \not\perp V_i$ for all $f \in W_i$, $i \in I$.*

Moreover, if $\{W_i\}_{i\in I}$ is a minimal frame of subspaces with respect to $\{v_i\}_{i\in I}$ for $\mathcal{H}$, then $\{S_{W,v}^{-1/2} W_i\}_{i\in I}$ is an orthogonal family of subspaces in $\mathcal{H}$.

PROOF. Suppose that (2) holds and towards a contradiction assume that there exists $i \in I$ and $0 \neq f = \sum_{j\in I, j\neq i} g_j \in W_i$ with $g_j \in W_j$. By (2), we have $g_j \perp V_i$ for all $j \neq i$, hence $f \perp V_i$, but this is a contradiction. Thus (1) follows.

To prove the opposite direction suppose that $\{W_i\}_{i\in I}$ is minimal. For each $i \in I$, let P_i denote the orthogonal projection onto $\overline{\text{span}}_{j\in I, j\neq i}\{W_j\}$. Let $i \in I$ and let V_i be defined by $V_i = (I - P_i)\mathcal{H}$ for all $i \in I$. By the definition of V_i, we have $W_j \perp V_i$ for all $j \neq i$. Towards a contradiction assume that there exists $f \in W_i$ with $\langle f, g\rangle = 0$ for all $g \in V_i$. Then $f \in P_i\mathcal{H}$ and so $W_i \cap P_i\mathcal{H} \neq \{0\}$, which is a contradiction.

For the moreover part, let $\{v_i e_{ij}\}_{j\in J_i}$ be an orthonormal basis for W_i for each $i \in I$. By Proposition 3.17, $S_{W,v}$ is the frame operator for $\{v_i e_{ij}\}_{i\in I, j\in J_i}$. Since $\{W_i\}_{i\in I}$ is minimal, Lemma 4.2 implies that $\{v_i e_{ij}\}_{i\in I, j\in J_i}$ is a minimal frame for $\mathcal{H}$ and hence is a Riesz basis for $\mathcal{H}$. Thus $\{S_{W,v}^{-1/2} v_i e_{ij}\}_{i\in I, j\in J_i}$ is an orthonormal sequence in $\mathcal{H}$. Since we have

$$S_{W,v}^{-1/2} W_i = \overline{\text{span}}_{j\in J_i}\{S_{W,v}^{-1/2} v_i e_{ij}\},$$

it follows that $\{S_{W,v}^{-1/2} W_i\}_{i\in I, j\in J_i}$ is an orthogonal sequence in $\mathcal{H}$. □

The next definition transfers the definition of Riesz bases and exact sequences to families of subspaces in a canonical way. The so-called Riesz decomposition will share most properties with Riesz bases. However it will turn out that being an exact frame of subspaces is strictly weaker than being a Riesz decomposition, a fact which differs from the situation in abstract frame theory.

DEFINITION 4.4. We call a frame of subspaces $\{W_i\}_{i\in I}$ with respect to some family of weights for $\mathcal{H}$ a *Riesz decomposition* of $\mathcal{H}$, if for every $f \in \mathcal{H}$ there is a unique choice of $f_i \in W_i$ so that $f = \sum_{i\in I} f_i$. A frame of subspaces with respect to some family of weights is *exact*, if it ceases to be a frame of subspaces once one element is deleted.

LEMMA 4.5. *Let $\{W_i\}_{i\in I}$ be a family of subspaces in $\mathcal{H}$, and for each $i \in I$ let $\{e_{ij}\}_{j\in J_i}$ be an orthonormal basis for W_i.*

(1) *The following conditions are equivalent.*
    (a) *$\{W_i\}_{i\in I}$ is a Riesz decomposition of $\mathcal{H}$.*
    (b) *$\{e_{ij}\}_{i\in I, j\in J_i}$ is a Riesz basis for $\mathcal{H}$.*
    (c) *$\{e_{ij}\}_{i\in I, j\in J_i}$ is an unconditional basis for $\mathcal{H}$.*

(2) *Let* $\{W_i\}_{i\in I}$ *be a frame of subspaces with respect to* $\{v_i\}_{i\in I}$ *for* $\mathcal{H}$. *If* $\{v_ie_{ij}\}_{i\in I,j\in J_i}$ *is an exact frame for* $\mathcal{H}$, *then also* $\{W_i\}_{i\in I}$ *is an exact frame of subspaces with respect to* $\{v_i\}_{i\in I}$ *for* $\mathcal{H}$. *The opposite implication is not valid.*

PROOF. First we prove (1). The equivalence (b) $\Leftrightarrow$ (c) follows immediately from the fact that $\{v_ie_{ij}\}_{i\in I,j\in J_i}$ is bounded and that a Schauder basis is a Riesz basis if and only if it is a bounded unconditional basis. The implication (b) $\Rightarrow$ (a) is obvious. To prove (a) $\Rightarrow$ (c) assume that $\{e_{ij}\}_{i\in I,j\in J_i}$ is not an unconditional basis. Hence there exist $f\in\mathcal{H}$ and sequences $\{c_{ij}\}_{i\in I,j\in J_i}$ and $\{d_{ij}\}_{i\in I,j\in J_i}$ with $f=\sum_{i\in I,j\in J_i}c_{ij}e_{ij}=\sum_{i\in I,j\in J_i}d_{ij}e_{ij}$ such that $c_{i_0j_0}\neq d_{i_0j_0}$ for some $i_0\in I$, $j_0\in J_{i_0}$. By construction $\{e_{i_0j}\}_{j\in J_{i_0}}$ is an orthonormal basis for W_{i_0}, hence $\sum_{j\in J_{i_0}}c_{i_0j}e_{i_0j}\neq\sum_{j\in J_{i_0}}d_{i_0j}e_{i_0j}$, which implies that $\{W_i\}_{i\in I}$ is not a Riesz decomposition.

To prove (2) suppose that $\{W_i\}_{i\in I}$ is a frame of subspaces with respect to $\{v_i\}_{i\in I}$ for $\mathcal{H}$. By Theorem 3.2, $\{v_ie_{ij}\}_{i\in I,j\in J_i}$ is a frame for $\mathcal{H}$. If this is exact, then, by definition, deleting one element $v_{i_0}e_{i_0j_0}$ does not leave a frame. Thus also $\{v_ie_{ij}\}_{i\in I\setminus\{i_0\},j\in J_i}$ does not form a frame. Applying Theorem 3.2 once more yields the first claim. The fact that the opposite implication is not valid is demonstrated by Example 4.7. $\square$

The next theorem is the analog of a well-known result in abstract frame theory (see [**9**, Theorem 6.1.1]), only the role of exact frames of subspaces differs from the frame situation.

THEOREM 4.6. *Let* $\{W_i\}_{i\in I}$ *be a frame of subspaces with respect to* $\{v_i\}_{i\in I}$ *for* $\mathcal{H}$. *Then the following conditions are equivalent.*

(1) $\{W_i\}_{i\in I}$ *is a Riesz decomposition of* $\mathcal{H}$.
(2) $\{W_i\}_{i\in I}$ *is minimal.*
(3) *The synthesis operator* $T_{W,v}$ *is one-to-one.*
(4) *The analysis operator* $T^*_{W,v}$ *is onto.*

Moreover, if $\{W_i\}_{i\in I}$ *is a Riesz decomposition of* $\mathcal{H}$, *then it is also an exact frame of subspaces for* $\mathcal{H}$. *The opposite implication is not valid.*

PROOF. First note that (3) $\Leftrightarrow$ (4) is always true for operators on a Hilbert space. Moreover, it is obvious that (1) implies (3).

Next we prove (4) $\Rightarrow$ (1). By Theorem 3.12, $T^*_{W,v}$ is an isomorphism. Therefore if it is onto, then it is invertible. Hence, $T_{W,v}$ is invertible. This implies that for every $f\in\mathcal{H}$ there exists a $\{f_i\}_{i\in I}\in\left(\sum_{i\in I}\oplus W_i\right)_{l^2}$ so that

$$f=T_{W,v}(\{f_i\}_{i\in I})=\sum_{i\in I}v_if_i.$$

If we have $f=\sum_{i\in I}f_i=\sum_{i\in I}g_i$ with $\{f_i\},\{g_i\}\in\left(\sum_{i\in I}\oplus W_i\right)_{l^2}$, then it follows that $T_{W,v}(\{v_i^{-1}f_i\}_{i\in I})=T_{W,v}(\{v_i^{-1}g_i\}_{i\in I})$. Since $T_{W,v}$ is one-to-one, $\{v_i^{-1}f_i\}_{i\in I}=\{v_i^{-1}g_i\}_{i\in I}$ and so $f_i=g_i$ for all $i\in I$. This shows the equivalence of (1), (3), and (4).

It remains to prove that (1) is equivalent to (2). If $\{W_i\}_{i\in I}$ is not a Riesz decomposition of $\mathcal{H}$, there exists an element $f\in\mathcal{H}$ and $f_i,g_i\in W_i$, $i\in I$ with $f_{i_0}\neq g_{i_0}$ for some $i_0\in I$ and $f=\sum_{i\in I}f_i=\sum_{i\in I}g_i$. It follows $0\neq g_{i_0}-f_{i_0}=\sum_{i\in I,i\neq i_0}(f_i-g_i)$ and $g_{i_0}-f_{i_0}\in W_{i_0}$. This proves

$$g_{i_0}-f_{i_0}\in W_{i_0}\cap\overline{\operatorname{span}}_{i\in I,i\neq i_0}\{W_i\},$$

which implies that $\{W_i\}_{i\in I}$ is not minimal.

To prove the converse implication assume that $\{W_i\}_{i\in I}$ is not minimal. Then, for some $i_0 \in I$, there exists $0 \neq f = \sum_{i\in I, i\neq i_0} f_i \in W_{i_0}$ with $f_i \in W_i$. Hence

$$0 = \sum_{i\in I, i\neq i_0} (f_i - f) = \sum_{i\in I} 0.$$

Thus $\{W_i\}_{i\in I}$ is not a Riesz decomposition of $\mathcal{H}$.

To prove the moreover part, suppose $\{W_i\}_{i\in I}$ is a frame of subspaces with respect to $\{v_i\}_{i\in I}$ for $\mathcal{H}$ and a Riesz decomposition of $\mathcal{H}$. We will prove that this implies that $\{W_i\}_{i\in I}$ is exact. For this, fix some $i_0 \in I$. Since $T^*_{W,v}$ is onto, there exists an element $f \in \mathcal{H}$ such that $\pi_{W_{i_0}}(f) \neq 0$, but $\pi_{W_i}(f) = 0$ for all $i \neq i_0$. Thus

$$\sum_{i\in I} v_i^2 \|\pi_{W_i}(f)\|^2 = v_{i_0}^2 \|\pi_{W_{i_0}}(f)\|^2.$$

Therefore it is not possible to delete W_{i_0} from the frame of subspaces yet leave a frame of subspaces. Since i_0 was chosen arbitrarily, the claim follows.

The fact that the opposite implication is not valid is demonstrated by Example 4.7. □

Note that we could have also proven the equivalence of (1) and (2) and the moreover part of the previous result by using Lemma 4.2, Lemma 4.5, and [**9**, Theorem 6.1.1]. We have chosen to add the extended proof here in order to enlighten the use of frames of subspaces.

Next we give an example for the different role exactness plays in the situation of families of subspaces.

Example 4.7. Let $\{e_i\}_{i\in\mathbb{Z}}$ be an orthonormal basis for some Hilbert space $\mathcal{H}$ and define the subspaces W_1, W_2 by

$$W_1 = \overline{\text{span}}_{i\geq 0}\{e_i\} \quad \text{and} \quad W_2 = \overline{\text{span}}_{i\leq 0}\{e_i\}.$$

Then $\{W_1, W_2\}$ is a frame of subspaces with respect to weights $\{v_1, v_2\}$ with $v_1 = v_2 =: v > 0$, since

$$v_1\|\pi_{W_1}(f)\|^2 + v_2\|\pi_{W_2}(f)\|^2 = \sum_{i\in\mathbb{Z}} v\,|\langle f, e_i\rangle|^2 + v\,|\langle f, e_0\rangle|^2 = v\|f\|^2 + v\,|\langle f, e_0\rangle|^2$$

and

$$v\|f\|^2 \leq v\|f\|^2 + v\,|\langle f, e_0\rangle|^2 \leq 2v\|f\|^2.$$

It is also exact, since when we delete one subspace the remaining one does not span the space. But it is not a Riesz decomposition, because we can write the element e_0 as $e_0 = e_0 + 0$ and $e_0 = 0 + e_0$. Thus the decomposition is not unique. Also observe that the sequence $\{ve_i\}_{i\geq 0} \cup \{ve_i\}_{i\leq 0}$ is a frame, but is not exact.

We conclude this subsection by mentioning that orthonormal bases of subspaces are special cases of Riesz decompositions.

Corollary 4.8. *If $\{W_i\}_{i\in I}$ is an orthonormal basis of subspaces for $\mathcal{H}$, then it is also a Riesz decomposition of $\mathcal{H}$.*

Proof. This follows immediately from the definition of a Riesz decomposition and Proposition 3.23. □

5. Several constructions

In this section we will discuss several constructions concerning frames of subspaces, frames, and Riesz frames. Recall that in addition to what follows we are already equipped with some constructions by Theorem 3.2, Corollary 3.21, and Lemma 4.5.

5.1. Constructions of frames of subspaces. Dealing with Bessel families of subspaces is important, since there are easy ways to turn such a family into a frame of subspaces. One way is to just add the subspace $W_0 = \mathcal{H}$ to the family. Another more careful method is the following one: Take any orthonormal basis for $\mathcal{H}$ and partition its elements into the subspaces W_i, $i \in I$. Then add the subspaces spanned by the remaining elements to the Bessel family. This yields a frame of subspaces.

Using the synthesis operator $T_{W,v}$, we obtain a characterization of Bessel sequences of subspaces.

PROPOSITION 5.1. *Let $\{W_i\}_{i\in I}$ be a family of subspaces of $\mathcal{H}$, and let $\{v_i\}_{i\in I}$ be a family of weights. Then the following conditions are equivalent.*

(1) *$\{W_i\}_{i\in I}$ is a Bessel sequence of subspaces with respect to $\{v_i\}_{i\in I}$ for $\mathcal{H}$.*
(2) *The synthesis operator $T_{W,v}$ is bounded and linear.*

PROOF. First suppose that (1) holds. Then Lemma 3.9 shows that the series in the definition of the synthesis operator $T_{W,v}$ converges unconditionally. Moreover, we have

$$\sum_{i\in I} v_i^2 \|\pi_{W_i}(f)\|^2 \le B\|f\|^2.$$

By definition of the analysis operator $T^*_{W,v}$,

$$\|T^*_{W,v}(f)\|^2 = \sum_{i\in I} v_i^2 \|\pi_{W_i}(f)\|^2. \tag{5.1}$$

Since $\{W_i\}_{i\in I}$ is a Bessel sequence of subspaces with respect to $\{v_i\}_{i\in I}$, this implies that $T^*_{W,v}$ is bounded. Hence also $T_{W,v}$ is bounded, which shows (2).

If (2) holds, then also $T^*_{W,v}$ is a bounded operator. This fact together with (5.1) yields (1). □

One possible application for frames of subspaces is to the problem of classifying those $g \in L^2(\mathbb{R})$ and $0 < a, b \le 1$ so that (g, a, b) yields a Gabor frame (see Example 5.4 below). This is an exceptionally deep problem even in the case of characteristic functions [**6, 20**]. But we have simple classifications of when $\{e^{2\pi imbt}g(t)\}_{m\in\mathbb{Z}}$ is a frame sequence in $L^2(\mathbb{R})$ and when $\{e^{2\pi imbt}g(t-na)\}_{m,n\in\mathbb{Z}}$ has dense span in $L^2(\mathbb{R})$. By our results, this family will be a Gabor frame for $L^2(\mathbb{R})$ if and only if $\{W_n\}_{n\in\mathbb{Z}}$ is a frame of subspaces where W_n is the closed linear span of $\{e^{2\pi imbt}g(t-na)\}_{m\in\mathbb{Z}}$.

For some applications, we would like to take a frame for $\mathcal{H}$ and divide it into subsets so that the closed linear span of these subsets is a frame of subspaces for $\mathcal{H}$. This is not always possible. But the next proposition shows that one of the needed inequalities will always hold.

PROPOSITION 5.2. *Let $\{f_j\}_{j\in J}$ be a frame for $\mathcal{H}$ with frame bounds A and B. Let $\{J_i\}_{i\in I}$ be a partition of the indexing set J, and for all $i \in I$ let W_i denote the* *closed linear span of* $\{f_j\}_{j\in J_i}$. *Then for all* $f \in \mathcal{H}$ *we have*

$$\frac{A}{B}\|f\|^2 \le \sum_{i\in I} \|\pi_{W_i}(f)\|^2.$$

Hence, if $|I| < \infty$, *then* $\{W_i\}_{i\in I}$ *is a* 1*-uniform frame of subspaces for* $\mathcal{H}$.

PROOF. We compute

$$A\|f\|^2 \le \sum_{j\in J} |\langle f, f_j\rangle|^2 = \sum_{i\in I}\sum_{j\in J_i} |\langle f, f_j\rangle|^2 = \sum_{i\in I}\sum_{j\in J_i} |\langle \pi_{W_i}(f), f_j\rangle|^2.$$

Recall that if a family of vectors is a B-Bessel family then every subfamily is also B-Bessel. Thus

$$\sum_{i\in I}\sum_{j\in J_i} |\langle \pi_{W_i}(f), f_j\rangle|^2 \le \sum_{i\in I} B\|\pi_{W_i}(f)\|^2.$$

This proves the first claim.

If $|I| < \infty$, then we have

$$\sum_{i\in I} \|\pi_{W_i}(f)\|^2 \le |I| \cdot \|f\|^2.$$

Hence in this case $\{W_i\}_{i\in I}$ is always a frame of subspaces for $\mathcal{H}$, in particular a 1-uniform frame of subspaces. □

An easy way to obtain a frame of subspaces is provided by the next result.

PROPOSITION 5.3. *Let* $\{f_j\}_{j\in J}$ *be a frame for* $\mathcal{H}$, *let* $J = J_1 \cup \ldots \cup J_n$ *be a finite partition of* J, *and let* $\{v_i\}_{i=1}^n$ *be a family of weights. Then* $\{W_i\}_{i=1}^n$ *is a frame of subspaces with respect to* $\{v_i\}_{i=1}^n$ *for* $\mathcal{H}$, *where* $W_i = \overline{\mathrm{span}}_{j\in J_i}\{f_j\}$.

PROOF. Let $f \in \mathcal{H}$. Obviously, $\|\pi_{W_i}(f)\|^2 \le \|f\|^2$ for all $1 \le i \le n$, which implies that

$$\sum_{i=1}^n v_i^2 \|\pi_{W_i}(f)\|^2 \le \max_{i=1,\ldots,n}\{v_i^2\} \cdot \|f\|^2.$$

Thus $\{W_i\}_{i=1}^n$ is a Bessel sequence of subspaces with respect to $\{v_i\}_{i=1}^n$. The lower bound follows from an application of Proposition 5.2. That is, for any $f \in \mathcal{H}$ we have

$$\frac{A}{B}\|f\|^2 \le \sum_{i\in I} \|\pi_{W_i}(f)\|^2 \le \frac{1}{\max_{i=1,\ldots,n}\{v_i^2\}} \sum_{i\in I} v_i^2\|\pi_{W_i}(f)\|^2.$$

□

This partition of the frame elements is not always a partition into frame sequences. Let us consider the case of Gabor systems. In the following example we will show that a large class of Gabor systems can be written as a frame of subspaces. Moreover, we can characterize those Gabor atoms, for which this partition is a partition into frame sequences.

EXAMPLE 5.4. For each $a \in \mathbb{R}$, let the unitary operators E_a, T_a on $L^2(\mathbb{R})$ be defined by

$$E_a f(x) = e^{2\pi i a x} f(x) \quad \text{and} \quad T_a f(x) = f(x-a).$$

Given a function $g \in L^2(\mathbb{R})$ and $a, b > 0$, the Gabor system determined by g and a, b is defined by

$$\mathcal{G}(g, a, b) = \{E_{ma}T_{nb}g : m, n \in \mathbb{Z}\}.$$

Let

$$Z : L^2(\mathbb{R}) \to L^2([0,1)^2), \quad Zf(x,y) = \sum_{k\in\mathbb{Z}} f(x+k)e^{2\pi iky}.$$

denote the Zak transform on $L^2(\mathbb{R})$ (compare [**19**]).

Let $h \in L^2(\mathbb{R})$ and $q \in \mathbb{N}$. In the following we will consider some Gabor system $\mathcal{G}(h,a,b)$ with $a,b > 0$, $ab = \frac{1}{q}$. Using a metaplectic transform it can be shown that this system is unitarily equivalent to $\mathcal{G}(g, \frac{1}{q}, 1)$ for some $g \in L^2(\mathbb{R})$ [**16**, Proposition 9.4.4], hence it suffices to consider this system. Now the Gabor system $\mathcal{G}(g, \frac{1}{q}, 1)$ in turn can be decomposed using the partition

$$\mathcal{G}(g, \tfrac{1}{q}, 1) = \bigcup_{j=0}^{q-1} \{E_{\frac{1}{q}(mq+j)}T_n g\}_{m,n\in\mathbb{Z}}. \tag{5.2}$$

By Proposition 5.3, the set of the subspaces $W_j := \overline{\text{span}}_{m,n\in\mathbb{Z}}\{E_{\frac{1}{q}(mq+j)}T_n g\}$, $j = 0,\ldots,q-1$ is indeed a frame of subspaces.

In a second step we will investigate, whether the sequences $\{E_{\frac{1}{q}(mq+j)}T_n g\}_{m,n\in\mathbb{Z}}$ are frame sequences. It will turn out that this will not happen unless the Zak transform is discontinuous.

We first observe that the following conditions are equivalent.

(1) The sequence $\{E_{\frac{1}{q}(mq+j)}T_n g\}_{m,n\in\mathbb{Z}}$ is a frame sequence for each $0 \le j \le q-1$.

(2) There exist $0 < A \le B < \infty$ such that

$$A \le |Zg(x,y)|^2 \le B \quad \text{for almost all } (x,y) \in [0,1)^2 \backslash V,$$

where $V = \{(x,y) \in [0,1)^2 : Zg(x,y) = 0\}$.

This can be proven as follows. First notice that since

$$E_{\frac{1}{q}}(\{E_{\frac{1}{q}(mq+j)}T_n g\}_{m,n\in\mathbb{Z}}) = \{E_{\frac{1}{q}(mq+j+1)}T_n g\}_{m,n\in\mathbb{Z}} \quad \text{for all } 0 \le j < q-1$$

and

$$E_{\frac{1}{q}}(\{E_{\frac{1}{q}(mq+q-1)}T_n g\}_{m,n\in\mathbb{Z}}) = \{E_{\frac{1}{q}(mq)}T_n g\}_{m,n\in\mathbb{Z}} = \{E_m T_n g\}_{m,n\in\mathbb{Z}},$$

the fact that $E_{\frac{1}{q}}$ is a unitary operator implies that condition (1) holds if and only if $\{E_m T_n g\}_{m,n\in\mathbb{Z}}$ is a frame sequence. Since $Z : L^2(\mathbb{R}) \to L^2([0,1)^2)$ is an isomorphism [**19**] and it is an easy calculation to show that $Z(E_m T_n g)(x,y) = E_m(x)E_n(y)Zg(x,y)$, condition (1) holds if and only if $\{E_m(x)E_n(y)Zg(x,y)\}_{m,n\in\mathbb{Z}}$ is a frame sequence. Since $\{E_m E_n\}_{m,n\in\mathbb{Z}}$ is an orthonormal basis for $L^2([0,1)^2)$, for each $f \in L^2([0,1)^2)$ we obtain

$$\sum_{m,n\in\mathbb{Z}} |\langle f, E_m E_n Zg\rangle|^2 = \left\|f \cdot \overline{Zg}\right\|^2.$$

This implies that (1) is equivalent to

$$A\|f\|^2 \le \left\|f \cdot \overline{Zg}\right\|^2 \le B\|f\|^2 \quad \text{for all } f \in \overline{\text{span}}_{m,n\in\mathbb{Z}}\{E_m(x)E_n(y)Zg(x,y)\},$$

which holds if and only if

$$A\|f \cdot Zg\|^2 \le \left\|f \cdot |Zg|^2\right\|^2 \le B\|f \cdot Zg\|^2 \quad \text{for all } f \in L^2([0,1)^2).$$

It is easy to check that this is equivalent to (2), which proves the claim.

Finally, we consider $g \in L^2(\mathbb{R})$ with Zg being continuous. By **[19]**, this implies that Zg has a zero. Hence condition (2) can never be fulfilled. This shows that the sequences $\{E_{\frac{1}{q}(mq+j)}T_n g\}_{m,n\in\mathbb{Z}}$ can only be frame sequences, if the Zak transform Zg is discontinuous.

5.2. Constructions of frames and Riesz frames. If we have a frame for $\mathcal{H}$, using a frame of subspaces for $\mathcal{H}$ we can construct new frames for $\mathcal{H}$ from these components.

PROPOSITION 5.5. *Let $\{W_i\}_{i\in I}$ be a frame of subspaces with respect to $\{v_i\}_{i\in I}$ for $\mathcal{H}$ and let $\{f_j\}_{j\in J}$ be a frame for $\mathcal{H}$. Then there exist $A, B > 0$ so that $\{\pi_{W_i} S_{W,v}^{-1}(f_j)\}_{j\in J}$ is a frame for W_i with frame bounds A and B for each $i \in I$. Hence $\{\pi_{W_i} S_{W,v}^{-1}(f_j)\}_{i\in I, j\in J}$ is also a frame for $\mathcal{H}$.*

PROOF. Since $S_{W,v}^{-1}$ is an invertible operator on $\mathcal{H}$ and $\{f_j\}_{j\in J}$ is a frame for $\mathcal{H}$, we have that $\{S_{W,v}^{-1} f_j\}_{j\in J}$ is a frame for $\mathcal{H}$ with frame bounds A and B. Therefore $\{\pi_{W_i} S_{W,v}^{-1}(f_j)\}_{j\in J}$ is a frame for W_j with frame bounds A and B for every $i \in I$. Since $\{W_i\}_{i\in I}$ is a frame of subspaces for $\mathcal{H}$, we have that $\{\pi_{W_i} S_{W,v}^{-1}(f_j)\}_{i\in I, j\in J}$ is a frame for $\mathcal{H}$ by Theorem 3.2. $\square$

To construct Riesz frames for $\mathcal{H}$ we first need to give an analog definition for families of subspaces.

DEFINITION 5.6. We call a frame of subspaces $\{W_i\}_{i\in I}$ a *Riesz frame of subspaces with respect to $\{v_i\}_{i\in I}$*, if there exist constants $C, D > 0$ so that every subfamily $\{W_i\}_{i\in J}$ with $J \subset I$ is a frame of subspaces with respect to $\{v_i\}_{i\in J}$ for its closed linear span with frame bounds C and D.

First we may ask whether subfamilies of a frame of subspaces are automatically frames of subspaces for their closed linear spans. The following example shows that this is not always the case.

EXAMPLE 5.7. In general, if $\{W_i\}_{i\in I}$ is a 1-uniform frame of subspaces and $J \subset I$, then $\{W_i\}_{i\in J}$ need not be a frame of subspaces for its closed linear span. For example, let $\{e_i\}_{i=1}^{\infty}$ be an orthonormal basis for $\mathcal{H}$ and for each $i \in I$ define the subspaces W_i^1, W_i^2, and W_i^3 by

$$W_i^1 = \text{span}\{e_{2i} + \tfrac{1}{i} e_{2i+1}\}, \quad W_i^2 = \text{span}\{e_{2i}\}, \quad \text{and} \quad W_i^3 = \text{span}\{e_{2i+1}\}.$$

Then it is easily checked that $\{W_i^1, W_i^2, W_i^3\}_{i=1}^{\infty}$ is a frame of subspaces for $\mathcal{H}$. Also observe that $\overline{\text{span}}_{i=1,\ldots,\infty}\{W_i^1, W_i^2\} = \mathcal{H}$. Since for all positive integers i we have

$$\pi_{W_i^1}(e_{2i+1}) = \frac{1}{i\sqrt{1+\frac{1}{i^2}}}(e_{2i} + \tfrac{1}{i} e_{2i+1}) \quad \text{and} \quad \pi_{W_i^2}(e_{2i+1}) = 0,$$

it follows that $\{W_i^1, W_i^2\}_{i=1}^{\infty}$ is not a frame of subspaces for its closed linear span.

Using a Riesz frame of subspaces and Riesz frames for the single subspaces, we can construct a Riesz frame for $\mathcal{H}$ by just taking all elements of the Riesz frames.

PROPOSITION 5.8. *Let $\{W_i\}_{i\in I}$ be a Riesz frame of subspaces with respect to $\{v_i\}_{i\in I}$ for $\mathcal{H}$, and let $\{f_{ij}\}_{j\in I_i}$ be a Riesz frame for W_i with Riesz frame bounds A and B for all $i \in I$. Then $\{v_i f_{ij}\}_{i\in I, j\in I_i}$ is a Riesz frame for $\mathcal{H}$. Also, for any $J \subset I$, $\{W_j\}_{j\in J}$ is a Riesz frame of subspaces with respect to $\{v_i\}_{i\in I}$ for its closed linear span.*

PROOF. Let C and D be the Riesz frame of subspaces bounds for $\{W_i\}_{i\in I}$. For every $i \in I$ choose $J_i \subset I_i$ and define $\widetilde{W}_i$ by

$$\widetilde{W}_i = \overline{\operatorname{span}}_{j\in J_i}\{f_{ij}\} \subset W_i.$$

Let $f \in \overline{\operatorname{span}}_{i\in I, j\in J_i}\{f_{ij}\}$. Then we have

$$\sum_{i\in I, j\in J_i} |\langle f, v_i f_{ij}\rangle|^2 = \sum_{i\in I} v_i^2 \sum_{j\in J_i} |\langle \pi_{\widetilde{W}_i}(f), f_{ij}\rangle|^2 \le \sum_{i\in I} B v_i^2 \|\pi_{\widetilde{W}_i}(f)\|^2 \le BD\|f\|^2.$$

Concerning the lower bounds, we compute

$$\sum_{i\in I, j\in J_i} |\langle f, v_i f_{ij}\rangle|^2 = \sum_{i\in I} v_i^2 \sum_{j\in J_i} |\langle \pi_{\widetilde{W}_i}(f), f_{ij}\rangle|^2 \ge \sum_{i\in I} A v_i^2 \|\pi_{\widetilde{W}_i}(f)\|^2 \ge CA\|f\|^2.$$

□

6. Harmonic frames of subspaces

Harmonic frames of subspaces are a special case of frames of subspaces, which are equipped with a natural structure and which occur in several situations, e.g., in Gabor analysis or in wavelet analysis in the form of multiresolution analysis.

6.1. The finite case. We start by giving the definition of a harmonic frame of subspaces for a finite family of subspaces.

DEFINITION 6.1. A frame of subspaces $\{W_i\}_{i\in I}$ with respect to $\{v_i\}_{i\in I}$ for $\mathcal{H}$ is a *finite harmonic frame of subspaces with respect to* $\{v_i\}_{i\in I}$, if $I = \{0, \ldots, N-1\}$, $N \in \mathbb{N}$ and there exists a unitary operator U on $\mathcal{H}$ so that

$$UW_{N-1} = W_0 \quad \text{and} \quad UW_i = W_{i+1} \quad \text{for all } 0 \le i \le N-2.$$

If $\{W_i\}_{i\in I}$ is a uniform Parseval frame of subspaces, $UW_{N-1} = W_0$ follows automatically as the following proposition shows. This result equals the corresponding result in the frame situation [**7**, Theorem 4.1].

THEOREM 6.2. *Let $\{W_i\}_{i\in I}$, $|I| = N$, be a uniform Parseval frame of subspaces such that there exists a unitary operator U on $\mathcal{H}$ so that*

$$UW_i = W_{i+1} \quad \textit{for all } 0 \le i \le N-2.$$

Then

$$UW_{N-1} = W_0.$$

PROOF. Without loss of generality we can assume that $v_i = 1$ for all $i \in I$. Let $\{g_j\}_{j\in J}$ be an orthonormal basis for W_0. By hypothesis, $\{U^i g_j\}_{j\in J}$ is an orthonormal basis for W_i for all $0 \le i \le N-1$. Now let $f \in \mathcal{H}$. By Proposition 3.23 and since $\{W_i\}_{i\in I}$ was assumed to be uniform, we have

$$f = \sum_{i=0}^{N-1} \sum_{j\in J} \langle f, U^i g_j\rangle U^i g_j. \tag{6.1}$$

Applying U, this leads to

$$\begin{aligned}
Uf &= \sum_{i=0}^{N-1} \sum_{j \in J} \langle Uf, U^i g_j \rangle\, U^i g_j \\
&= \sum_{i=0}^{N-1} \sum_{j \in J} \langle f, U^{i-1} g_j \rangle\, U^i g_j \\
&= U \left[\sum_{i=0}^{N-1} \sum_{j \in J} \langle f, U^{i-1} g_j \rangle\, U^{i-1} g_j \right] \\
&= U \left[\sum_{i=-1}^{N-2} \sum_{j \in J} \langle f, U^{i} g_j \rangle\, U^{i} g_j \right].
\end{aligned}$$

Since U is unitary and by (6.1), we obtain

$$f = \sum_{i=-1}^{N-2} \sum_{j \in J} \langle f, U^i g_j \rangle\, U^i g_j = \sum_{i=0}^{N-1} \sum_{j \in J} \langle f, U^i g_j \rangle\, U^i g_j.$$

This implies

$$\sum_{j \in J} \langle f, U^{-1} g_j \rangle\, U^{-1} g_j = \sum_{j \in J} \langle f, U^{N-1} g_j \rangle\, U^{N-1} g_j.$$

Now we apply U, which yields

$$\sum_{j \in J} \langle f, U^{-1} g_j \rangle\, g_j = \sum_{j \in J} \langle f, U^{N-1} g_j \rangle\, U^{N} g_j.$$

Using $U^{-1} f$ instead of f gives

$$\sum_{j \in J} \langle f, g_j \rangle\, g_j = \sum_{j \in J} \langle f, U^{N} g_j \rangle\, U^{N} g_j,$$

which shows that $\pi_{W_0} = \pi_{\overline{\text{span}}_{j \in J}\{U^N g_j\}} = \pi_{UW_{N-1}}$. This completes the proof. □

The following result is [**7**, Theorem 4.2 and Theorem 4.3], which we add for completeness in a reformulated version and with a proof using our results.

PROPOSITION 6.3. *Let $\varphi \in \mathcal{H}$ and let V be a unitary operator on $\mathcal{H}$ such that $\{V^j\varphi\}_{j=0}^{M-1}$ is a uniform Parseval frame sequence. Define the subspace W_0 by $W_0 = \text{span}_{j=0,\ldots,M-1}\{V^j\varphi\}$. Further let U be a unitary operator on $\mathcal{H}$. Then the following conditions are equivalent.*

(1) *$\{U^iV^j\varphi\}_{i=0,\ j=0}^{L-1,\ M-1}$ is a uniform Parseval frame for $\mathcal{H}$.*
(2) *$\{U^iW_0\}_{i=0}^{L-1}$ is a uniform Parseval frame of subspaces for $\mathcal{H}$.*

PROOF. This follows immediately from Theorem 3.2 with setting $W_i = U^iW_0$ for all $i = 1, \ldots, L-1$. □

We conclude this subsection by giving an example for this type of frames of subspaces.

EXAMPLE 6.4. Let $a \in \mathbb{R}$ and $g \in L^2(\mathbb{R})$ be such that $\{E_{am}T_ng\}_{m,n\in\mathbb{Z}}$ is a Parseval Gabor frame. Fix some $N \in \mathbb{N}$ and define W_i, $0 \le i \le N-1$, by

$$W_i = \overline{\text{span}}_{m,n\in\mathbb{Z}}\{E_{a(mN+i)}T_ng\}.$$

Then $\{W_i\}_{i=0}^{N-1}$ is a finite harmonic frame of subspaces. To see this let $U := E_a$. Then

$$UW_{N-1} = W_0 \quad \text{and} \quad UW_i = W_{i+1} \quad \text{for all } 0 \le i \le N-2.$$

Also,

$$U^i(E_{amN}T_ng) = E_{a(mN+i)}T_ng.$$

Hence the sequences $\{E_{a(mN+i)}T_ng\}_{m,n\in\mathbb{Z}}$ are even unitarily equivalent to each other.

Notice that this construction generalizes the one in Example 5.4.

6.2. The infinite case. We can also define a harmonic frame of subspaces for an infinite family of subspaces.

DEFINITION 6.5. A frame of subspaces $\{W_i\}_{i\in I}$ with respect to $\{v_i\}_{i\in I}$ for $\mathcal{H}$ is an *infinite harmonic frame of subspaces with respect to* $\{v_i\}_{i\in I}$, if $I = \mathbb{Z}$ and there exists a unitary operator U on $\mathcal{H}$ so that

$$UW_i = W_{i+1} \quad \text{for all } i \in I.$$

An interesting example for this type of frame of subspaces is the generalized frame multiresolution analysis in the sense of Papadakis [**21**], which generalizes the classical multiresolution analysis.

EXAMPLE 6.6. Here we consider the generalized frame multiresolution analysis in the sense of Papadakis [**21**], whose approach includes all classical multiresolution analysis (MRAs) in one and higher dimensions as well as the FMRAs of Benedetto and Li [**2**].

Let $\mathcal{H}$ be a Hilbert space, $U : \mathcal{H} \to \mathcal{H}$ be a unitary operator, and let G be a unitary abelian group acting on $\mathcal{H}$. A sequence $\{V_i\}_{i\in\mathbb{Z}}$ of closed subspaces of $\mathcal{H}$ is a *generalized frame multiresolution analysis* of $\mathcal{H}$, if it satisfies the following properties.

(1) $V_i \subseteq V_{i+1}$ for all $i \in \mathbb{Z}$.
(2) $V_i = U^i(V_0)$ for all $i \in \mathbb{Z}$.
(3) $\bigcap_i V_i = \{0\}$ and $\overline{\bigcup_i V_i} = \mathcal{H}$.
(4) There exists a countable subset B of V_0 such that the set $G(B) = \{g\phi : g \in G, \phi \in B\}$ is a frame for V_0.

Now $\{W_i\}_{i\in\mathbb{Z}}$ is defined by $W_0 := V_1 \cap V_0^\perp$ and $W_i := U^i(W_0)$ for every $i \in \mathbb{Z}$. Then we have

$$W_i = U^i(V_1 \cap V_0^\perp) = U^i(V_1) \cap U^i(V_0)^\perp = V_{i+1} \cap V_i^\perp.$$

In the case $\mathcal{H} = L^2(\mathbb{R})$, $G = \{T_k : k \in \mathbb{Z}\}$, B containing only one element, and $U = D_2$ being the dilation operator $D_2 f(t) = \sqrt{2}f(2t)$ this definition reduces to the well-known MRA.

In the general situation we have the following relations:

(i) If $\{V_i\}_{i\in\mathbb{Z}}$ is a GFMRA of some Hilbert space $\mathcal{H}$, then $\{W_i\}_{i\in\mathbb{Z}}$ is an infinite 1-uniform harmonic Parseval frame of subspaces, since $\mathcal{H} = \bigoplus_{i\in\mathbb{Z}} W_i$, with unitary operator U.

(ii) Let $\{W_i\}_{i\in\mathbb{Z}}$ be an infinite 1-uniform harmonic Parseval frame of subspaces with unitary operator denoted by U. By Proposition 3.23, we have $\mathcal{H} = \bigoplus_{i\in\mathbb{Z}} W_i$. Then we can define $\{V_i\}_{i\in\mathbb{Z}}$ by

$$V_i = \bigoplus_{m\leq i-1} W_m.$$

Then (1)-(3) are obviously satisfied. Therefore $\{W_i\}_{i\in\mathbb{Z}}$ is a GFMRA if and only if (4) is satisfied.

Acknowledgments

The majority of the research for this paper was performed while the second author was visiting the Departments of Mathematics at the University of Missouri and Washington University in St. Louis. This author thanks these departments for their hospitality and support during these visits.

The second author is indebted to Damir Bakić, Ilya Krishtal, Demetrio Labate, Guido Weiss, and Ed Wilson for helpful discussions.

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