Duality principles, localization of frames, and Gabor theory

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ABSTRACT

The theory of localized frames is a recently introduced concept with broad implications to frame theory in general, as well as to the special cases of Gabor and wavelet frames. Using the new notion of an R-dual sequence associated with a Bessel sequence, we derive several duality principles concerning localization in abstract frame theory. As applications of our results we prove a duality principle of localization of Gabor systems in the spirit of the Ron-Shen duality principle, and obtain a Janssen representation for general frame operators.

Keywords: Bessel sequence, duality principle, frame, Gabor system, Janssen representation, localization, R-dual sequence

1. INTRODUCTION

Hilbert space frame theory has seen an explosion of new results and applications in recent years. One of the most important concrete realizations of frames are the Gabor frames which form the basis for time-frequency analysis. Recently, many results from Gabor theory have found their manifestation in abstract frame theory and further led to new results for other frames such as wavelet frames. Let $g \in L^2(\mathbb{R})$ and $a, b > 0$, then the associated *Gabor system* $\mathcal{G}(g, a, b)$ is defined by

$$\mathcal{G}(g, a, b) = \{M_{mb}T_{na}g : m, n \in \mathbb{Z}\},$$

where $T_{na}f(x) = f(x - na)$ and $M_{mb}f(x) = e^{2\pi imbx}f(x)$. Thus the elements of $\mathcal{G}(g, a, b)$ are indexed by the lattice $a\mathbb{Z} \times b\mathbb{Z}$. The lattice $\frac{1}{b}\mathbb{Z} \times \frac{1}{a}\mathbb{Z}$ is then called the *adjoint lattice* and $\mathcal{G}(g, \frac{1}{b}, \frac{1}{a})$ is referred to as the *associated Gabor system with respect to the adjoint lattice*. Several of the most fascinating results in Gabor theory are duality principles between a Gabor system and its associated Gabor system with respect to the adjoint lattice, e.g., the Ron-Shen duality principle[12] and the Wexler-Raz biorthogonality relations.[13]

Quite recently these ideas have been generalized to abstract frame theory.[4] For a Bessel sequence in a separable Hilbert space an associated sequence, the so-called R-dual sequence, has been introduced. The relation between those two sequences turned out to be an important tool for deriving duality principles in abstract frame theory. In fact, almost all duality principles from Gabor theory have analogues in abstract frame theory in terms of a Bessel sequence and its R-dual sequence. For several special cases, e.g., the case of tight Gabor frames, it is already known that the associated Gabor system with respect to the adjoint lattice coincides with the R-dual sequence of the Gabor system itself. However, it is still an open problem whether for any Gabor system $\mathcal{G}(g, a, b)$ the associated Gabor system with respect to the adjoint lattice $\mathcal{G}(g, \frac{1}{b}, \frac{1}{a})$ coincides with the R-dual sequence of $\mathcal{G}(g, a, b)$.

The concept of localization of a frame was introduced in 2004 by Balan, Casazza, Heil, and Landau[1] and independently by Gröchenig[10] and enjoys increasing attention. The basic idea for introducing this new notion was to formulate the conditions which constitute a "good frame". Powerful machinery in abstract frame theory[6, 9] is being developed using this concept with several important implications to Gabor and wavelet theory.

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The goal of this paper is to study the relation between localization of a Bessel sequence and its R-dual sequence. These results are then on the one hand used to derive new results concerning localization of a Gabor system and its associated Gabor system with respect to the adjoint lattice, and on the other hand to generalize useful tools from Gabor theory to abstract frame theory.

First we give a brief review of the definitions and notations from frame theory and the theory of localization in Section 2, and state some results, which will be used in the sequel. Section 3 deals with localization properties of a Bessel sequence and its R-dual sequence. In particular, we will study the relation between localization of a Bessel sequence and localization of its R-dual sequence and show several characterizations of localization of a Bessel sequence in terms of its R-dual sequence. In the last section we will elaborate on two applications of the results derived in Section 3. First, in Subsection 4.1, we will characterize localization of a Gabor system in terms of localization of the associated Gabor system with respect to the adjoint lattice, thus deriving a duality principle in the spirit of the Ron-Shen duality principle and the Wexler-Raz biorthogonality relations. In Subsection 4.2 we will go the other way round and generalize a useful tool from Gabor theory to the abstract frame setting. One of the main tools for studying a Gabor frame operator is its Janssen representation,[11] which is formulated in terms of the associated Gabor system with respect to the adjoint lattice. Using the notion of an R-dual sequence and results from Section 3 we derive a representation analogous to Janssen's for a general frame operator in this subsection.

2. DEFINITIONS AND BACKGROUND

First, we briefly recall the definitions and basic properties of frames. Then we state the definition of ℓ_1-localization and some known results, which will be used in the sequel.

For the remainder let $\mathcal{H}$ be a separable Hilbert space.

2.1. Frame theory

A family $g = \{g_n\}_{n\in\mathbb{Z}}$ is a *frame* for $\mathcal{H}$, if there exist $0 < A \le B < \infty$ such that for all $f \in \mathcal{H}$,

$$A\,\|f\|_2^2 \le \sum_{n\in\mathbb{Z}} |\langle f, g_n\rangle|^2 \le B\,\|f\|_2^2. \tag{1}$$

The constants A and B are called a *lower* and *upper frame bound*. If we assume only the upper inequality in (1) then $\{g_n\}_{n\in\mathbb{Z}}$ called a *Bessel sequence*. A frame is *tight*, if $A = B$. The *frame operator* $S_g f = \sum_{n\in\mathbb{Z}}\langle f, g_n\rangle g_n$ associated with a frame $\{g_n\}_{n\in\mathbb{Z}}$ is a bounded, invertible, and positive mapping of $\mathcal{H}$ onto itself. This provides the frame decomposition

$$f = S_g^{-1}S_g f = \sum_{n\in\mathbb{Z}}\langle f, g_n\rangle \tilde{g}_n = \sum_{n\in\mathbb{Z}}\langle f, \tilde{g}_n\rangle g_n,$$

where $\tilde{g}_n = S_g^{-1}g_n$. The family $\{\tilde{g}_n\}_{n\in\mathbb{Z}}$ is also a frame for $\mathcal{H}$, called the *canonical dual frame* of $\{g_n\}_{n\in\mathbb{Z}}$.

Let $\{g_n\}_{n\in\mathbb{Z}}$ be a Bessel sequence in $\mathcal{H}$ and let $\{e_n\}_{n\in\mathbb{Z}}$ and $\{h_n\}_{n\in\mathbb{Z}}$ be orthonormal bases for $\mathcal{H}$. For each $n \in \mathbb{Z}$, define w_n^g by

$$w_n^g = \sum_{m\in\mathbb{Z}}\langle g_m, e_n\rangle h_m.$$

Then $\{w_n^g\}_{n\in\mathbb{Z}}$ is called the *R-dual sequence (Riesz-dual sequence) for* $\{g_n\}_{n\in\mathbb{Z}}$ *with respect to* $\{e_n\}_{n\in\mathbb{Z}}$ *and* $\{h_n\}_{n\in\mathbb{Z}}$.

The following result[4] is a kind of duality relation between the Bessel sequence $\{g_n\}_{n\in\mathbb{Z}}$ and its R-dual sequence $\{w_n^g\}_{n\in\mathbb{Z}}$.

LEMMA 2.1. *Let* $\{g_n\}_{n\in\mathbb{Z}}$ *be a Bessel sequence for* $\mathcal{H}$, *and let* $\{e_n\}_{n\in\mathbb{Z}}$ *and* $\{h_n\}_{n\in\mathbb{Z}}$ *be orthonormal bases for* $\mathcal{H}$. *Then, for all* $n \in \mathbb{Z}$,

$$g_n = \sum_{m\in\mathbb{Z}}\langle w_m^g, h_n\rangle e_m.$$

In particular, this result shows that $\{g_n\}_{n\in\mathbb{Z}}$ is the R-dual sequence for $\{w_n^g\}_{n\in\mathbb{Z}}$ with respect to $\{h_n\}_{n\in\mathbb{Z}}$ and $\{e_n\}_{n\in\mathbb{Z}}$.

For more details on frame theory we refer to the survey article by Casazza[3] and the book by Christensen.[5] An extensive study of the theory of R-dual sequences can be found in the paper by Casazza, Kutyniok, and Lammers.[4]

2.2. Localization

In this paper we will use the definition of ℓ_1-localization given by Balan, Casazza, Heil, and Landau.[1] But we will focus on the special case of ℓ_1-localization with respect to an orthonormal basis or ℓ_1-self-localization.

DEFINITION 2.2. *Let $\{g_n\}_{n\in\mathbb{Z}}$ be a sequence in $\mathcal{H}$.*

1. $\{g_n\}_{n\in\mathbb{Z}}$ *is* ℓ_1-localized with respect to $\{e_n\}_{n\in\mathbb{Z}}$, *if*

$$\sum_{m\in\mathbb{Z}} \sup_{n\in\mathbb{Z}} |\langle g_n, e_{m+n}\rangle| < \infty.$$

Equivalently, there must exist some $\{r_n\}_{n\in\mathbb{Z}} \in \ell_1(\mathbb{Z})$ such that, for all $m, n \in \mathbb{Z}$,

$$|\langle g_m, e_n\rangle| \le r_{m-n}.$$

2. $\{g_n\}_{n\in\mathbb{Z}}$ *is* ℓ_1-self-localized, *if*

$$\sum_{m\in\mathbb{Z}} \sup_{n\in\mathbb{Z}} |\langle g_n, g_{m+n}\rangle| < \infty.$$

Equivalently, there must exist some $\{r_n\}_{n\in\mathbb{Z}} \in \ell_1(\mathbb{Z})$ such that, for all $m, n \in \mathbb{Z}$,

$$|\langle g_m, g_n\rangle| \le r_{m-n}.$$

The following two results will be used throughout the paper. They are weaker versions of results from Balan, Casazza, Heil, and Landau.[1, 2]

LEMMA 2.3. *If $\{e_n\}_{n\in\mathbb{Z}}$ is an orthonormal basis and $\{g_n\}_{n\in\mathbb{Z}}$ is ℓ_1-localized with respect to it, then $\{g_n\}_{n\in\mathbb{Z}}$ is ℓ_1-self-localized.*

LEMMA 2.4. *Let $\{g_n\}_{n\in\mathbb{Z}}$ be a frame with canonical dual frame denoted by $\{\tilde{g}_n\}_{n\in\mathbb{Z}}$. If $\{g_n\}_{n\in\mathbb{Z}}$ is ℓ_1-self-localized, then so is $\{\tilde{g}_n\}_{n\in\mathbb{Z}}$.*

For further details on the theory of localization of frames we refer to the papers by Balan, Casazza, Heil, and Landau.[1, 2]

3. LOCALIZATION OF A BESSEL SEQUENCE AND ITS R-DUAL SEQUENCE

In this section we study localization properties of a Bessel sequence and its R-dual sequence. In particular, we will investigate ℓ_1-localization and ℓ_1-self-localization of a Bessel sequence in terms of its R-dual sequence.

Throughout this section let $g = \{g_n\}_{n\in\mathbb{Z}}$ be a Bessel sequence for $\mathcal{H}$, and fix two orthonormal bases $\{e_n\}_{n\in\mathbb{Z}}$ and $\{h_n\}_{n\in\mathbb{Z}}$ for $\mathcal{H}$. The R-dual sequence associated with $\{g_n\}_{n\in\mathbb{Z}}$ with respect to $\{e_n\}_{n\in\mathbb{Z}}$ and $\{h_n\}_{n\in\mathbb{Z}}$ is denoted by $w^g = \{w_n^g\}_{n\in\mathbb{Z}}$. If $\{g_n\}_{n\in\mathbb{Z}}$ is a frame for $\mathcal{H}$, S_g denotes its frame operator.

Concerning ℓ_1-localization of a Bessel sequence and its R-dual sequence we obtain the following duality principle.

PROPOSITION 3.1. *The following conditions are equivalent.*

1. *$\{g_n\}_{n\in\mathbb{Z}}$ is ℓ_1-localized with respect to $\{e_n\}_{n\in\mathbb{Z}}$.*
2. *$\{w_n^g\}_{n\in\mathbb{Z}}$ is ℓ_1-localized with respect to $\{h_n\}_{n\in\mathbb{Z}}$.*

Proof. For all $m, n \in \mathbb{Z}$, we have

$$|\langle w_m^g, h_n \rangle| = |\sum_{k \in \mathbb{Z}} \langle g_k, e_m \rangle \langle h_k, h_n \rangle| = |\langle g_n, e_m \rangle|.$$

The result now follows from the definition of ℓ_1-localization. □

Next we investigate both sequences with respect to ℓ_1-self-localization. The following proposition provides us with several equivalent conditions for a sequence to be ℓ_1-localized with respect to an orthonormal basis in terms of its R-dual sequence.

PROPOSITION 3.2. *Let $\{g_n\}_{n\in\mathbb{Z}}$ be a frame for $\mathcal{H}$. Then the following conditions are equivalent.*

1. *$\{g_n\}_{n\in\mathbb{Z}}$ is ℓ_1-self-localized.*
2. *$\{S_{w^g}^{\frac{1}{2}}(h_n)\}_{n\in\mathbb{Z}}$ is ℓ_1-self-localized.*
3. *$\{S_{w^g}(h_n)\}_{n\in\mathbb{Z}}$ is ℓ_1-localized with respect to $\{h_n\}_{n\in\mathbb{Z}}$.*

Proof. Using Lemma 2.1, for all $m, n \in \mathbb{Z}$, we have

$$\begin{aligned} |\langle g_m, g_n \rangle| &= |\sum_{k\in\mathbb{Z}} \sum_{l\in\mathbb{Z}} \langle h_n, w_k^g \rangle \langle e_l, e_k \rangle \langle w_l^g, h_m \rangle| \\ &= |\sum_{k\in\mathbb{Z}} \langle h_n, w_k^g \rangle \langle w_k^g, h_m \rangle| \\ &= |\langle S_{w^g}(h_n), h_m \rangle| \\ &= |\langle S_{w^g}^{\frac{1}{2}}(h_n), S_{w^g}^{\frac{1}{2}}(h_m) \rangle|. \end{aligned}$$

From this, the claim follows immediately. □

Using this result we can now characterize ℓ_1-localization of a frame with respect to some orthonormal basis in terms of ℓ_1-localization of its frame operator applied to the same orthonormal basis. This fact will turn out to become important in Subsection 4.2.

PROPOSITION 3.3. *Let $\{g_n\}_{n\in\mathbb{Z}}$ be a frame for $\mathcal{H}$. If $\{g_n\}_{n\in\mathbb{Z}}$ is ℓ_1-localized with respect to $\{e_n\}_{n\in\mathbb{Z}}$, then $\{S_g(e_n)\}_{n\in\mathbb{Z}}$ is ℓ_1-localized with respect to $\{e_n\}_{n\in\mathbb{Z}}$.*

Proof. This follows from Proposition 3.1, Lemma 2.3, Proposition 3.2, and Lemma 2.1. □

The following result studies ℓ_1-localization of $\{S_g(e_n)\}_{n\in\mathbb{Z}}$ and $\{S_{\tilde{g}}(e_n)\}_{n\in\mathbb{Z}}$, where $\tilde{g} = \{\tilde{g}_n\}_{n\in\mathbb{Z}}$ denotes the dual frame.

LEMMA 3.4. *Let $\{g_n\}_{n\in\mathbb{Z}}$ be a frame with its dual frame denoted by $\{\tilde{g}_n\}_{n\in\mathbb{Z}}$. If $\{S_g(e_n)\}_{n\in\mathbb{Z}}$ is ℓ_1-localized with respect to $\{e_n\}_{n\in\mathbb{Z}}$, then $\{S_{\tilde{g}}(e_n)\}_{n\in\mathbb{Z}}$ is ℓ_1-localized with respect to $\{e_n\}_{n\in\mathbb{Z}}$.*

Proof. Suppose that $\{S_g(e_n)\}_{n\in\mathbb{Z}}$ is ℓ_1-localized with respect to $\{e_n\}_{n\in\mathbb{Z}}$. First observe that, since $S_g^{1/2}$ is a self-adjoint operator, we can write

$$|\langle S_g^{1/2}(e_m), S_g^{1/2}(e_n) \rangle| = |\langle S_g(e_m), e_n \rangle|.$$

Thus $\{S_g^{1/2}(e_n)\}_{n\in\mathbb{Z}}$ is ℓ_1-self-localized. Since $S_g^{1/2}$ is an invertible operator, the sequence $\{S_g^{1/2}(e_n)\}_{n\in\mathbb{Z}}$ is a Riesz basis. Moreover, the computation

$$\begin{aligned} f &= S_g^{-1/2} S_g^{1/2}(f) \\ &= \sum_{n\in\mathbb{Z}} \langle S_g^{1/2}(f), e_n \rangle S_g^{-1/2}(e_n) \\ &= \sum_{n\in\mathbb{Z}} \langle f, S_g^{1/2}(e_n) \rangle S_g^{-1/2}(e_n) \end{aligned}$$

shows that the dual Riesz basis of $\{S_g^{1/2}(e_n)\}_{n\in\mathbb{Z}}$ is $\{S_g^{-1/2}(e_n)\}_{n\in\mathbb{Z}}$. Hence Lemma 2.4 implies that the sequence $\{S_g^{-1/2}(e_n)\}_{n\in\mathbb{Z}}$ is ℓ_1-self-localized. Now, by the same argument as above, it follows that $\{S_g^{-1}(e_n)\}_{n\in\mathbb{Z}}$ is ℓ_1-localized with respect to $\{e_n\}_{n\in\mathbb{Z}}$. Since $S_g^{-1} = S_{\tilde{g}}$, the claim is proven. □

4. APPLICATIONS

In this section we will state two interesting consequences of our results from Section 3. First we will characterize ℓ_1-localization of a Gabor system in terms of ℓ_1-localization of the associated Gabor system with respect to the adjoint lattice, which can be regarded as a new duality principle. Secondly, we will prove a version of the Janssen representation[11] for general frames.

4.1. Gabor systems with respect to the adjoint lattice

Interestingly, none of the major contributions to the theory of localization of Gabor systems[2, 6, 9, 10] addresses the question of the relation between localization of a Gabor system and the associated Gabor system with respect to the adjoint lattice. But it is fair to assume that there exists a strong relation between both, since there exist many duality principles in Gabor theory for those pairs of systems, e.g., the Ron-Shen duality principle[12] and the Wexler-Raz biorthogonality relations.[13]

In the case of a Gabor system we have the following result[4] concerning the exact form of the R-dual sequence.

PROPOSITION 4.1. *Let $g \in L^2(\mathbb{R})$ and $a, b > 0$ be such that $\mathcal{G}(g, a, b)$ is a tight frame for $L^2(\mathbb{R})$. Then, for each bounded and compactly supported orthonormal basis $\{e_n\}_{n\in\mathbb{Z}}$ for $L^2(\mathbb{R})$, there exists a unitary operator $U : L^2(\mathbb{R}) \to L^2(\mathbb{R})$ so that $\mathcal{G}(g, \frac{1}{b}, \frac{1}{a})$ is the R-dual sequence of $\mathcal{G}(g, a, b)$ with respect to $\{e_n\}_{n\in\mathbb{Z}}$ and $\{U(e_n)\}_{n\in\mathbb{Z}}$.*

This leads us to a relation between localization of a Gabor system and localization of the associated Gabor system with respect to the adjoint lattice in the setting of tight frames.

PROPOSITION 4.2. *Let $g \in L(\mathbb{R})$ and $a, b > 0$ be such that $\mathcal{G}(g, a, b)$ is a tight frame for $L^2(\mathbb{R})$. Then, for each bounded and compactly supported orthonormal basis $\{e_n\}_{n\in\mathbb{Z}}$ for $L^2(\mathbb{R})$, there exists a unitary operator $U : L^2(\mathbb{R}) \to L^2(\mathbb{R})$ so that the following conditions are equivalent.*

1. *$\mathcal{G}(g, a, b)$ is ℓ_1-localized with respect to $\{e_n\}_{n\in\mathbb{Z}}$.*
2. *$\mathcal{G}(g, \frac{1}{b}, \frac{1}{a})$ is ℓ_1-localized with respect to $\{U(e_n)\}_{n\in\mathbb{Z}}$.*

Proof. Applying Proposition 3.1 to Proposition 4.1 yields the claim. ☐

Whether Proposition 4.1 holds in full generality, i.e., for each Gabor system, is still unknown. Therefore we pose the following conjecture.[4]

CONJECTURE 4.3. *Let $g \in L(\mathbb{R})$ and $a, b > 0$. Then there exist orthonormal bases $\{e_n\}_{n\in\mathbb{Z}}$ and $\{h_n\}_{n\in\mathbb{Z}}$ for $L^2(\mathbb{R})$ so that $\mathcal{G}(g, \frac{1}{b}, \frac{1}{a})$ coincides with the R-dual sequence of $\mathcal{G}(g, a, b)$ with respect to $\{e_n\}_{n\in\mathbb{Z}}$ and $\{h_n\}_{n\in\mathbb{Z}}$.*

If the conjecture turns out to be true, Proposition 4.2 would hold for general Gabor systems, i.e., condition (i) and (ii) from Proposition 4.2 would be equivalent for any Gabor system.

4.2. The Janssen Representation for general frames

Let $g \in L^2(\mathbb{R})$ and $a, b > 0$. A Gabor system $\mathcal{G}(g, a, b)$ satisfies *condition (A)*, if

$$\sum_{m,n\in\mathbb{Z}} |\langle g, M_{\frac{m}{a}} T_{\frac{n}{b}} g\rangle| < \infty.$$

In this case, its frame operator possesses the Janssen representation[11]

$$S_{\mathcal{G}(g,a,b)}(f) = \frac{1}{ab} \sum_{m,n\in\mathbb{Z}} \langle g, M_{\frac{m}{a}} T_{\frac{n}{b}} g\rangle M_{\frac{m}{a}} T_{\frac{n}{b}}(f) \tag{2}$$

with absolute convergence in the operator norm. This symmetric representation is a very useful tool for studying a Gabor frame operator. In this subsection we will generalize this representation to the abstract frame setting.

Let $g = \{g_n\}_{n\in Z}$ be a frame for $\mathcal{H}$. In this general situation condition (A) is substituted by the condition that $\{S_g(e_n)\}_{n\in\mathbb{Z}}$ has to be ℓ_1-localized with respect to $\{e_n\}_{n\in\mathbb{Z}}$. In fact, we have the following relation.

Lemma 4.4. *Let $g \in L(\mathbb{R})$ and $a, b > 0$, and let $\{e_{mn}\}_{m,n\in\mathbb{Z}}$ be an orthonormal basis for $L^2(\mathbb{R})$. Furthermore, assume that Conjecture 4.3 is true. Then $\{S_{\mathcal{G}(g,a,b)}(e_{mn})\}_{m,n\in\mathbb{Z}}$ being ℓ_1-localized with respect to $\{e_{mn}\}_{m,n\in\mathbb{Z}}$ implies condition (A).*

Proof. For this, let $\{h_{mn}\}_{m,n\in\mathbb{Z}}$ be also an orthonormal basis for $L^2(\mathbb{R})$. For all $k, l \in \mathbb{Z}$, we compute

$$\begin{aligned}
|\langle g, M_{\frac{m}{a}} T_{\frac{n}{b}} g\rangle| &= |\langle w_{00}^{\mathcal{G}(g,a,b)}, w_{mn}^{\mathcal{G}(g,a,b)}\rangle| \\
&= |\sum_{i,j,k,l\in\mathbb{Z}} \langle M_{ib} T_{ja} g, e_{00}\rangle \langle h_{ij}, h_{kl}\rangle \langle e_{mn}, M_{kb} T_{la} g\rangle| \\
&= |\sum_{i,j\in\mathbb{Z}} \langle M_{ib} T_{ja} g, e_{00}\rangle \langle e_{mn}, M_{ib} T_{ja} g\rangle| \\
&= |\langle S_{\mathcal{G}(g,a,b)}(e_{mn}), e_{00}\rangle|.
\end{aligned}$$

If $\{S_{\mathcal{G}(g,a,b)}(e_{mn})\}_{m,n\in\mathbb{Z}}$ is ℓ_1-localized with respect to $\{e_{mn}\}_{m,n\in\mathbb{Z}}$, there exists some $\{r_{mn}\}_{m,n\in\mathbb{Z}} \in \ell_1(\mathbb{Z}^2)$ such that

$$\sum_{m,n\in\mathbb{Z}} |\langle S_{\mathcal{G}(g,a,b)}(e_{mn}), e_{00}\rangle| \leq \sum_{m,n\in\mathbb{Z}} |r_{mn}| < \infty.$$

Thus condition (A) is satisfied. □

We obtain the following generalization of the Janssen representation for general frames.

Theorem 4.5. *Let $\{g_n\}_{n\in\mathbb{Z}}$ be a frame for $\mathcal{H}$ and let $\{e_n\}_{n\in\mathbb{Z}}$ be an orthonormal basis for $\mathcal{H}$ for which $\{S_g e_n\}_{n\in\mathbb{Z}}$ is ℓ_1-localized with respect to $\{e_n\}_{n\in\mathbb{Z}}$. Assume there exist operators $\{V_n\}_{n\in\mathbb{Z}}$ on $\mathcal{H}$ so that*

1. *$V_n(S_g e_0) = S_g e_n$, for all $n \in \mathbb{Z}$.*
2. *There is a constant $B > 0$ so that for all $m \in \mathbb{Z}$ the set $\{V_n(e_m)\}_{n\in\mathbb{Z}}$ is a Bessel sequence with bound B.*

Then there exist bounded operators $\{W_n\}_{n\in\mathbb{Z}}$, which satisfy $\sup_{n\in\mathbb{Z}} \|W_n\| < \infty$, such that S_g has the Janssen representation

$$S_g(f) = \sum_{n\in\mathbb{Z}} \langle w_n^g, w_0^g\rangle W_n(f), \tag{3}$$

with absolute convergence in the operator norm.

Proof. We will do a computation with infinite series and later justify switching the order of summation.

$$\begin{aligned}
S_g(f) &= \sum_{m\in\mathbb{Z}} \langle f, e_m\rangle S_g(e_m) \\
&= \sum_{m\in\mathbb{Z}} \langle f, e_m\rangle V_m S_g(e_0) \\
&= \sum_{m\in\mathbb{Z}} \langle f, e_m\rangle V_m \left(\sum_{n\in\mathbb{Z}} \langle S_g(e_0), e_n\rangle e_n\right) \\
&= \sum_{n\in\mathbb{Z}} \langle S_g(e_0), e_n\rangle \sum_{m\in\mathbb{Z}} \langle f, e_m\rangle V_m(e_n) \\
&= \sum_{n\in\mathbb{Z}} \left[\sum_{k\in\mathbb{Z}} \langle e_0, g_k\rangle \langle g_k, e_n\rangle\right] \sum_{m\in\mathbb{Z}} \langle f, e_m\rangle V_m(e_n) \\
&= \sum_{n\in\mathbb{Z}} \langle w_n^g, w_0^g\rangle \sum_{m\in\mathbb{Z}} \langle f, e_m\rangle V_m(e_n).
\end{aligned}$$

Defining W_n by $W_n(f) = \sum_{m\in\mathbb{Z}} \langle f, e_m\rangle V_m(e_n)$ yields the claimed formula for (3).

Now we need to check convergence. Since $\{V_m(e_n)\}_{m\in\mathbb{Z}}$ is a Bessel sequence with bound B, we have for any finite set $I \subset \mathbb{Z}$,

$$\|\sum_{m\in I}\langle f, e_m\rangle V_m(e_n)\|^2 \leq B\sum_{m\in I}|\langle f, e_m\rangle|^2 = B\|f\|^2.$$

It follows that the series defining W_n are unconditionally convergent and $\|W_n\| \leq \sqrt{B}$, for all $n \in \mathbb{Z}$.

If $I, J \subset \mathbb{Z}$ are finite sets then from our above calculations,

$$\begin{aligned}
\|\sum_{n\in I}\langle w_n^g, w_0^g\rangle \sum_{m\in J}\langle f, e_m\rangle V_m(e_n)\| &= \|\sum_{n\in I}|\langle S_g(e_0), e_n\rangle \sum_{m\in J}\langle f, e_m\rangle V_m(e_n)\| \\
&\leq \left[\sum_{n\in I}|\langle S_g(e_0), e_n\rangle|\right]\|\sum_{m\in J}\langle f, e_m\rangle W_m(e_n)\| \\
&\leq \left[\sum_{n\in I}|\langle S_g(e_0), e_n\rangle|\right]\sqrt{B}\|f\|.
\end{aligned}$$

Since $\{S_g e_n\}_{n\in\mathbb{Z}}$ is ℓ_1-localized, it follows that our series in the construction of W_n are unconditionally convergent justifying our interchange of the order of summation.

Finally, for a fixed finite $I \subset \mathbb{Z}$ with $\{-N, -(N-1), \ldots, N\} \subset I$ we have:

$$\begin{aligned}
\|S_g - \sum_{n\in I}\langle w_n^g, w_0^g\rangle W_n\| &= \sup_{\|f\|=1}\|S_g(f) - \sum_{n\in I}\langle w_n^g, w_0^g\rangle W_n(f)\| \\
&= \sup_{\|f\|=1}\|\sum_{n\in I^c}\langle w_n^g, w_0^g\rangle W_n(f)\| \\
&\leq \left[\sum_{|n|\geq N}|\langle w_n^g, w_0^g\rangle|\right]\sup_{\|f\|=1}\sup_{|n|\geq N}\|W_n(f)\| \\
&\leq \left[\sum_{|n|\geq N}|\langle w_n^g, w_0^g\rangle|\right]\sqrt{B}.
\end{aligned}$$

Using Lemma 2.1 and Proposition 3.2, it follows that our operators converge to S_g unconditionally in the operator norm. $\square$

It is easy to get operators $\{V_n\}_{n\in\mathbb{Z}}$ satisfying condition 1. and 2. of Theorem 4.5. For example, just define $V_n(S_g e_0) = S_g(e_n)$ and define $V_n(f) = 0$ for all $f \in [S_g(e_0)]^\perp$. If P is the orthogonal projection onto the span of $S_g(e_0)$, then

$$V_n(e_m) = \|Pe_m\|S_g(e_n).$$

Hence, for all $m \in \mathbb{Z}$,

$$\{V_n(e_m)\}_{n\in\mathbb{Z}} = \{\|Pe_m\|S_g(e_n)\}_{n\in\mathbb{Z}}.$$

Since $\{S_g(e_n)\}_{n\in\mathbb{Z}}$ is a Riesz basis for the space, it follows that the families $\{V_n(e_m)\}_{n\in\mathbb{Z}}$ are all Bessel with the same Bessel bound.

A more interesting example is the following.

EXAMPLE 4.6. Let $\{g_n\}_{n\in\mathbb{Z}}$ and $\{e_n\}_{n\in\mathbb{Z}}$ satisfy the hypotheses of Theorem 4.5. Further, let $U : \mathcal{H} \to \mathcal{H}$ be the unitary shift operator $U(e_n) := e_{n+1}$, and let $V : \mathcal{H} \to \mathcal{H}$ be the operator defined by $V := S_g U S_g^{-1}$. Then we define the operators $\{V_n\}_{n\in\mathbb{Z}}$ on $\mathcal{H}$ by

$$V_n := V^n = S_g U^n S_g^{-1}.$$

Condition 1. of Theorem 4.5 is obviously fulfilled, since

$$V_n(S_g e_0) = V^n(S_g e_0) = S_g(U^n e_0) = S_g e_n \quad \text{for all } n \in \mathbb{Z}.$$

To check condition 2., let $(c_n)_{n\in\mathbb{Z}}$ be a sequence in $\ell_2(\mathbb{Z})$. Then there exists $f \in \mathcal{H}$ such that $c_n = \langle f, e_n\rangle$ for all $n \in \mathbb{Z}$. First we compute

$$\begin{aligned}
\|\sum_{n\in\mathbb{Z}} c_n V^n(e_m)\| &= \|\sum_{n\in\mathbb{Z}} \langle f, e_n\rangle V^n(e_m)\| \\
&= \|S_g\left(\sum_{n\in\mathbb{Z}} \langle f, e_n\rangle U^n(S_g^{-1}(e_m))\right)\| \\
&\leq \|S_g\|\|\sum_{n\in\mathbb{Z}} \langle f, e_n\rangle \sum_{k\in\mathbb{Z}} \langle S_g^{-1}(e_m), U^{-n}(e_k)\rangle e_k\| \\
&\leq \|S_g\|\,\|\sum_{n\in\mathbb{Z}} \langle f, e_n\rangle \sum_{k\in\mathbb{Z}} \langle S_g^{-1}(e_m), e_{k-n}\rangle e_k\|.
\end{aligned}$$

Since $S_g^{-1} = S_{\tilde{g}}$, Lemma 3.4 implies that $\{S_g^{-1}(e_m)\}_{m\in\mathbb{Z}}$ is ℓ_1-localized with respect to $\{e_m\}_{m\in\mathbb{Z}}$. Hence there exists $\{s_m\}_{m\in\mathbb{Z}} \in \ell_1(\mathbb{Z})$ such that $|\langle S_g^{-1}(e_n), e_m\rangle| \leq s_{n-m}$ for all $m, n \in \mathbb{Z}$. Therefore, for each $m \in \mathbb{Z}$, we have

$$\sup_{n\in\mathbb{Z}} \sum_{k\in\mathbb{Z}} |\langle S_g^{-1}(e_m), e_{k-n}\rangle| = \sup_{k\in\mathbb{Z}} \sum_{n\in\mathbb{Z}} |\langle S_g^{-1}(e_m), e_{k-n}\rangle| \leq \|s\|_1 < \infty.$$

Thus we can apply Schur's test[8] to the matrix $(A^n_{n,k})_{n,k\in\mathbb{Z}}$ defined by $A^m_{n,k} = \langle S_g^{-1}(e_m), e_{k-n}\rangle$, which yields

$$\begin{aligned}
&\|S_g\|\|\sum_{n\in\mathbb{Z}} \langle f, e_n\rangle \sum_{k\in\mathbb{Z}} \langle S_g^{-1}(e_m), e_{k-n}\rangle e_k\| \\
&\leq \|S_g\|\,\|s\|_1 \sqrt{\sum_{n\in\mathbb{Z}} |\langle f, e_n\rangle|^2} \\
&= \|s\|_1\,\|S_g\|\,\|f\|.
\end{aligned}$$

Therefore $\{V_n(e_m)\}_{n\in\mathbb{Z}}$ is a Bessel sequence with bound $\|s\|_1\,\|S_g\|$.

Now we consider when we can get the two Janssen representations for S_g and S_g^{-1} to be the same.

PROPOSITION 4.7. *Given the conditions in the theorem, the Janssen representations of S_g and S_g^{-1} are the same if and only if their corresponding sequences $\{V_n\}_{n\in\mathbb{Z}}$ from Theorem 4.5 are equal.*

Proof. Suppose we have two Janssen representations for S_g and S_g^{-1} given by the operators V_n and $\tilde{V}_n$ and the representations are the same. That is, $W_n = \tilde{W}_n$ for all $n \in \mathbb{Z}$. Then for all $k \in \mathbb{Z}$ we have

$$V_k(e_n) = \sum_{m\in\mathbb{Z}} \langle e_k, e_m\rangle V_m(e_n) = W_n(e_k) = \tilde{W}_n(e_k) = \tilde{V}_k(e_n).$$

Therefore, $V_k = \tilde{V}_k$ for all $k \in \mathbb{Z}$. The converse is obvious. □

The conditions in Proposition 4.7 are also easy to obtain. For example, if we assume (as we may) that $S_g \neq S_g^{-1}$ then for some $n \in \mathbb{Z}$ we have that $S_g(e_n) \neq S_g^{-1}(e_n)$. So reindex the basis taking e_n to e_0. That is, we may assume that $S_g(e_0) \neq S_g^{-1}(e_0)$. Then we may define our V_n by

$$V_n(S_g(e_0)) = S_g(e_n), \quad V_n(S_g^{-1}(e_0)) = S_g(e_n),$$

and V_n equals zero in the orthogonal complements of the span of these two vectors. As we saw in our earlier example, we now get that $\{V_n(e_m)\}_{n\in\mathbb{Z}}$ is a sum of two Riesz bases for the space and hence is still uniformly Bessel.

Let $k \in \mathbb{Z}$. We remark that we can substitute condition 1. of Theorem 4.5 by the condition that $V_n(S_g e_k) = S_g e_n$ for all $n \in \mathbb{Z}$, which leads to a Janssen representation (3), in which the reference element w_0^g is substituted by w_k^g.

Further, if we change the definition of the R-dual sequence $\{w_n^g\}_{n\in\mathbb{Z}}$ associated with some Bessel sequence $\{g_n\}_{n\in\mathbb{Z}}$ in $\mathcal{H}$ and two orthonormal bases $\{e_n\}_{n\in\mathbb{Z}}$ and $\{h_n\}_{n\in\mathbb{Z}}$ for $\mathcal{H}$ to

$$w_n^g = \sum_{m\in\mathbb{Z}} \langle e_n, g_m\rangle h_m \quad (n \in \mathbb{Z}),$$

then we obtain the following general form of the Janssen representation:

$$S_g(f) = \sum_{n\in\mathbb{Z}} \langle w_0^g, w_n^g\rangle W_n(f).$$

If one of the inner products in the Janssen representation is complex, i.e., we have $\langle w_0^g, w_n^g\rangle \neq \langle w_n^g, w_0^g\rangle$ for some $n \in \mathbb{Z}$, then only this approach would yield the same form of the inner products as in the classical Janssen representation (2).

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