Signal Analysis with Frame Theory and Persistent Homology

Holger Boche
TU München
Theresienstr. 90/IV (LTI)
80333 München, Germany
boche@tum.de

Mijail Guillemard
TU Berlin
Institut für Mathematik
10623 Berlin, Germany
guillemard@math.tu-berlin.de

Gitta Kutyniok
TU Berlin
Institut für Mathematik
10623 Berlin, Germany
kutyniok@math.tu-berlin.de

Friedrich Philipp
TU Berlin
Institut für Mathematik
10623 Berlin, Germany
philipp@math.tu-berlin.de

***Abstract*—A basic task in signal analysis is to characterize data in a meaningful way for analysis and classification purposes. Time-Frequency transforms are powerful strategies for signal decomposition, and important recent generalizations have been achieved in the setting of frame theory. In parallel recent developments, tools from algebraic topology, traditionally developed in purely abstract settings, have provided new insights in applications to data analysis. In this report, we investigate some interactions of these tools, both theoretically and with numerical experiments in order to characterize signals and their corresponding adaptive frames. We explain basic concepts in persistent homology as an important new subfield of computational topology, as well as formulations of time-frequency analysis in frame theory. Our objective is to use persistent homology for constructing topological signatures of signals in the context of frame theory for classification and analysis purposes. The main motivation for studying these interactions is to combine the strength of frame theory as a fundamental signal analysis methodology, with persistent homology as a novel perspective in data analysis.**

I. Introduction

Modern developments in signal processing have triggered important interactions between pure and applied mathematics. A basic example is given by new advances in time-frequency analysis and its generalizations to frame theory [3, 5], but another recent and major development illustrating a rich interplay between abstract ideas and practical applications is persistent homology [2], which in the last few years has become an important subfield of computational topology. In this report we continue our previous research [7] which introduced some strategies for integrating time-frequency analysis with persistent homology. Our focus now is on the usage of frame theory and dictionary learning algorithms for adaptive signal analysis. Our contribution is to investigate topological properties of frames constructed from dictionary learning algorithms (e.g. k-svd [1]). In particular, we are interested in using the stability of persistent diagrams as a conceptual link between frame transformations and persistent homology.

The outline of this report is as follows. We begin with a short overview of time-frequency analysis and frame theory, with a particular focus on voice transformations and how this setting is generalized in (continuous) frame theory by considering analysis operators $V : \mathcal{H} \to L^2(\mathcal{X})$. Here, $\mathcal{X}$ is a locally compact group for the case of voice transformations, and a general topological space in frame theory.

We then present persistent homology as a new important branch in data analysis which, given a point cloud data $X = \{x_i\}_{i=1}^m$, constructs a persistent diagram which can be seen as a topological fingerprint that fully encodes topological features of X. An important property we consider in this report is the stability of persistent diagrams under special topological and functional perturbations. Our goal is to combine these tools and relate the frame operator machinery with the persistence homology theory in order to design better adapted signal analysis strategies. We also present some components of numerical simulations that illustrate the interaction between our main topics: adapted dictionaries, their frames properties, and topological characterization using persistent diagrams.

II. Time-Frequency Analysis and Frame Theory

Given a Hilbert space $\mathcal{H}$ as, for instance, a functional space of signals $L^2(\mathbb{R})$, the basic strategy in time-frequency analysis is to segment a signal $f \in \mathcal{H}$ in smaller chunks $x_b = fg_b$, for g a window function, and $g_b(t) = g(t-b)$. This segmentation procedure allows to locally analyze the frequency behavior of f and its evolution in time. A generalization of this method can be described using a locally compact group G acting in a Hilbert space $\mathcal{H}$ (see [5]). This action is an irreducible and square integrable group representation, $\pi : G \to U(\mathcal{H})$, defined as a group homomorphism between G and $U(\mathcal{H})$, the group of unitary operators in $\mathcal{H}$. The basic transformation that is constructed with π is the *analysis operator* or the *voice transform*:

$$V_\psi : \mathcal{H} \to L^2(G), \qquad V_\psi(f)(x) = \langle f, \pi(x)\psi \rangle,$$

which maps each $f \in \mathcal{H}$ to a square integrable function $V_\psi f$ that "unfolds" the content of f in the setting provided by G. The fundamental property of V is to be a quasi isometry, as described with the formula:

$$\int_{\mathcal{G}} \langle f_1, \pi(x)\psi_1 \rangle \langle \pi(x)\psi_2, f_2 \rangle d\mu(x) = \langle f_1, f_2 \rangle \langle C\psi_1, C\psi_2 \rangle.$$

The two basic examples of voice transforms are given by Gabor analysis, and short time Fourier transforms, constructed with the Heisenberg group, and wavelet theory, which uses the affine group. The Heisenberg group is defined with $\mathcal{G} =$

$\mathbb{R}\times\mathbb{R}\times\mathbb{T}$, where the product is set as $(a_1, b_1, t_1)(a_2, b_2, t_2) = (a_1 + a_2, b_1 + b_2, t_1t_2e^{2\pi ib_1a_2})$. The corresponding representation is given by $(\pi(a, b, t)f)(y) = te^{2\pi ib(y-a)f(y-a)}$ for $f \in L^2(\mathbb{R}), y \in \mathbb{R}$. In the case of the affine group (sometimes denominated the $ax+b$ group) we have $G = \mathbb{R}\times\mathbb{R}^+\backslash\{0\}$ with the product given by $(b, a)(x, s) = (ax + b, as)$ and the corresponding representation is $(\pi(b, a)f)(y) = \frac{1}{\sqrt{a}}f\Big(\frac{y-b}{a}\Big)$.

A. Continuous and Discrete Frames

Despite the major role of the voice transform and its group representation background, in some applications it is too restrictive to assume the existence of a group G that parametrize the family of dictionary vectors $\{\pi(x)\psi\}_{x\in G}$. An important generalization of these procedures is frame theory which considers a family of vectors $\{\psi_x\}_{x\in\mathcal{X}}$ in a Hilbert space $\mathcal{H}$, where $\mathcal{X}$ is a topological space (see [3]). When $\mathcal{X}$ is discrete (e.g. $\mathcal{X} = \mathbb{N}$) or finite, this concept is a generalization of an orthogonal basis, and it provides powerful mechanisms for the analysis and synthesis of a signal $f \in \mathcal{H}$.

The main property required by a frame $\{\psi_x\}_{x\in\mathcal{X}} \subset \mathcal{H}$ is the stabilization of the analysis operator.

Definition 1. A set of vectors $\{\psi_x\}_{x\in\mathcal{X}} \subset \mathcal{H}$ in a Hilbert space $\mathcal{H}$ is *frame* if

$$A||f||^2 \leq ||Vf||^2 \leq B||f||^2, \quad \forall f \in H$$

for $0 < A \leq B < \infty$, the *lower and upper frame bounds*, and $V : \mathcal{H} \to L^2(\mathcal{X}), (Vf)(x) = \langle f, \psi_x\rangle$ is the *analysis operator*.

Reducing the difference between A and B improves the stability of V, and for the case of $A = B$, or $A = B = 1$, the resulting frame is denominated *tight frame* and *Parseval frame*, respectively. The corresponding synthesis operator $V^* : L^2(\mathcal{X}) \to V$, with $V^*((a_x)_{x\in\mathcal{X}}) = \int_{\mathcal{X}} a_x\psi_x\, d\mu(x)$ is defined with an adequate positive Radon measure μ, when $\mathcal{X}$ is a locally compact Hausdorff space (see [6]). The maps V^* and V are combined in the frame operator

$$S = V^*V : \mathcal{H} \to \mathcal{H}, Sf = \int_{\mathcal{X}} \langle f, \psi_x\rangle\psi_x d\mu(x),$$

which plays an important role due to the following property:

Theorem 1 ([3])**.** For a finite frame $\{\phi_i\}_{i=1}^m \subset \mathbb{R}^n$ with operator S having eigenvalues $\lambda_1 \geq \cdots \geq \lambda_n$, the eigenvalue λ_1 coincides with the optimal upper frame bound, and λ_n is the optimal lower frame bound.

As a consequence, we notice that the operator norm of S can be bounded by A and B, namely:

$$A \leq ||S||_{op} \leq B. \tag{1}$$

We remark that, as explained in [6], the continuous formulation of the frame properties can be related to a discretization problem. Here, the fundamental issue is to find conditions under which a discrete frame $\{\psi_{x_i}\}_{i\in I}$ (I discrete), can be constructed from a continuous frame $\{\psi_x\}_{x\in\mathcal{X}}$ ($\mathcal{X}$ a topological space). The key component is to use an adequate covering $\mathcal{U} = \{U_i\}_{i\in I}$ of $\mathcal{X}$, with selected elements $x_i \in U_i$, and properties based on the kernel $R(x, y) = \langle\psi_x, S^{-1}\psi_y\rangle, x, y \in \mathcal{X}$.

III. Persistent Homology

In order to shortly introduce the basic ideas in persistent homology, we first recall elementary ideas in simplicial homology. One of the simplest homology theories available is simplicial homology which translates topological data into an algebraic formulation. The fundamental objective is to compute qualitative properties of a topological space X, as the number of n-dimensional holes X has. The basic object to analyze is a (finite) *abstract simplicial complex* K, defined as a nonempty family of subsets of a vertex set $V = \{v_i\}_{i=1}^m$ with $\{v\} \in K$ if $v \in V$, and if $\alpha \in K, \beta \subseteq \alpha$, then $\beta \in K$. We define *faces* (or *simplices*) to be the elements of K, and their corresponding *dimension* will be their cardinality minus one.

In order to compute the number of holes of a given simplicial complex K, we translate its topological or combinatorial properties in the language of linear algebra. There are three basic steps in this procedure: first, we construct a family of free groups C_n, the *group of n-chains* defined as the formal combinations of k-dimensional faces with coefficients in a given group (or rings and fields in more specific cases). Secondly, we construct the *boundary operators* ∂_n, defined as homomorphisms (or more specifically linear maps) between the group of k-chains C_k. The homomorphism maps a face $\sigma = [p_0, \cdots p_n] \in C_n$ into C_{n-1} by

$$\partial_n\sigma = \sum_{k=0}^{n}(-1)^k[p_0, \cdots, p_{k-1}, p_{k+1}, \cdots p_n].$$

Finally, in the third step, we construct the *homology groups* defined as the quotients $H_k := \ker(\partial_k)/\mathrm{im}(\partial_{k+1})$. The main property is now the computation of the *Betti numbers*, which represent the amount of k-dimensional holes, and it corresponds to the rank of the homology groups, $\beta_k = \mathrm{rank}(H_k)$.

When considering these procedures for studying a point cloud data $X = \{x_i\}_{i=1}^m$, the major problem is the lack of a combinatorial structure in X encoding the topological interactions and similarities between the elements x_i. Persistent homology proposes a strategy for this problem by constructing the so called Vietoris Rips complexes, $R_\epsilon(X)$, defined with X as the vertex set, and setting the vertices $\sigma = \{x_0, \ldots, x_k\}$ to span a k-simplex of $R_\epsilon(X)$ if $d(x_i, x_j) \leq \epsilon$ for all $x_i, x_j \in \sigma$. The fundamental remark in the persistent homology for a finite point cloud data $X = \{x_i\}_{i=1}^m$ is that only a finite number of Vietoris Rips complexes are required to fully characterize the continuous family $\{R_\epsilon(X)\}_{\epsilon>0}$. Namely, there is only a finite number of non-homeomorphic simplicial complexes $K_1 \subset K_2 \subset \cdots \subset K_r$ (a so called *filtration*) that fully describe the set $\{R_\epsilon(X)\}_{\epsilon>0}$. Therefore, for a given ϵ there is an explicit K_i in the filtration $K_1 \subset K_2 \subset \cdots \subset K_r$ that is homeomorphic to $R_\epsilon(X)$.

In persistent homology, these constructions are formalized in a commutative diagram that combines the components we have discussed:

$$
\begin{array}{ccccccc}
\vdots & & \vdots & & \vdots & & \\
\downarrow \partial_3 & & \downarrow \partial_3 & & \downarrow \partial_3 & & \\
C_2^0 & \xrightarrow{f^0} & C_2^1 & \xrightarrow{f^1} & C_2^2 & \xrightarrow{f^2} & \dots \\
\downarrow \partial_2 & & \downarrow \partial_2 & & \downarrow \partial_2 & & \\
C_1^0 & \xrightarrow{f^0} & C_1^1 & \xrightarrow{f^1} & C_1^2 & \xrightarrow{f^2} & \dots \\
\downarrow \partial_1 & & \downarrow \partial_1 & & \downarrow \partial_1 & & \\
C_0^0 & \xrightarrow{f^0} & C_0^1 & \xrightarrow{f^1} & C_0^2 & \xrightarrow{f^2} & \dots \\
\downarrow \partial_0 & & \downarrow \partial_0 & & \downarrow \partial_0 & & \\
0 & \xrightarrow{f^0} & 0 & \xrightarrow{f^1} & 0 & \xrightarrow{f^2} & \dots
\end{array}
$$

Here, each column corresponds to the set of groups of k-chains, and the maps f^i that link each column are group homomorphisms induced by the inclusions $K_i \subset K_{i+1}$. The existence of the group homomorphisms f^i is a consequence of the functorial properties of the homology constructions, and this plays a major role, both in the theoretical and implementation aspects of these algorithms. We remark that the inclusions $K_i \subset K_j$ also induce corresponding group homomorphisms $f_n^{ij} : H_n(K_i) \to H_n(K_{i+j})$. With these tools, we have now the major ingredients for computing topological properties of the point cloud data.

Definition 2 (Persistent homology groups and persistent diagrams)**.** Given a dataset $X = \{x_i\}_{i=1}^m$ with a corresponding filtration $K_1 \subset \cdots \subset K_r$, we define the *persistent homology groups* as the image of f_n^{ij} for $f_n^{ij} : H_n(K_i) \to H_n(K_{i+j})$, which represent the homology classes born at i and still alive at $i + j$. The rank of these images $\beta_n^{ij} = \text{rank}(\text{Im} f_n^{ij})$ are the *persistent Betti numbers*. The *persistent diagram* ,D_X, of X considers pairs (i, j), $1 \leq i \leq j \leq r$ with the corresponding value β_n^{ij}.

A. Stability Properties

We now present an important component in the persistent homology toolbox denominated the *stability of persistent diagrams* [4]. In order to introduce this concept, we first introduce two basic concepts.

Definition 3 (Homological critical value and tame functions)**.** Let $f : X \to \mathbb{R}$ be a continuous function with X a topological space. An *Homological critical value* (or HCV) is a number $a \in \mathbb{R}$ for which the map induced by f

$$H_n(f^{-1}(] - \infty, a - \epsilon[)) \to H_n(f^{-1}(] - \infty, a + \epsilon[))$$

is not a (group) isomorphism, for all $\epsilon > 0$. A *tame function* is now defined to be a function $f : X \to \mathbb{R}$ that has only a finite number of HCV.

Theorem 2 (Stability of persistent diagrams)**.** Let X be a topological space with tame functions $f, g : X \to \mathbb{R}$, then we have

$$d_B(D_f, D_g) \leq ||f - g||_\infty \qquad (2)$$

with the bottleneck distance: $d_B(D_f, D_g) = \inf_\gamma \sup ||x - \gamma(x)||_\infty$, where we consider all bijections $\gamma : D_f \to D_g$. Here, we use $||p - g||_\infty = \max\{|p_1 - q_1|, |p_2 - q_2|\}$ for $p, q \in \mathbb{R}^2$.

IV. Frames and Persistent Diagrams

Our objective is to combine the core concepts in frame theory with persistent diagrams in order to integrate the strength of these different analysis tools. We describe a first elementary property that gives a theoretical hint on stability properties of persistent diagrams and frame analysis. This property specifies a stability property of the persistent diagrams when considering a frame decomposition $V(f) \in L^2(X)$, where X is the (continuous) parameter set of the frame $\{\psi_x\}_{x \in X}$.

Proposition 1. For two tame signals $f, g \in \mathcal{H}$ and a frame analysis operator $V : \mathcal{H} \to L^2(\mathcal{X})$ we have

$$d_B(Vf, Vg) \leq \sqrt{B}||f - g||_2.$$

Proof: This is a straightforward consequence of the Inequality 1 (the bounding of the norm of the frame operator) and the stability of the persistent diagrams described in the Inequality 2:

$$
\begin{aligned}
d_B(Vf, Vg) &\leq ||Vf - Vg||_2 \\
&\leq ||V||_{op}||f - g||_2 \\
&\leq \sqrt{||V^*V||_{op}}||f - g||_2 \qquad ||V||^2 = ||V^*V|| \\
&\leq \sqrt{||S||_{op}}||f - g||_2 \\
&\leq \sqrt{B}||f - g||_2.
\end{aligned}
$$

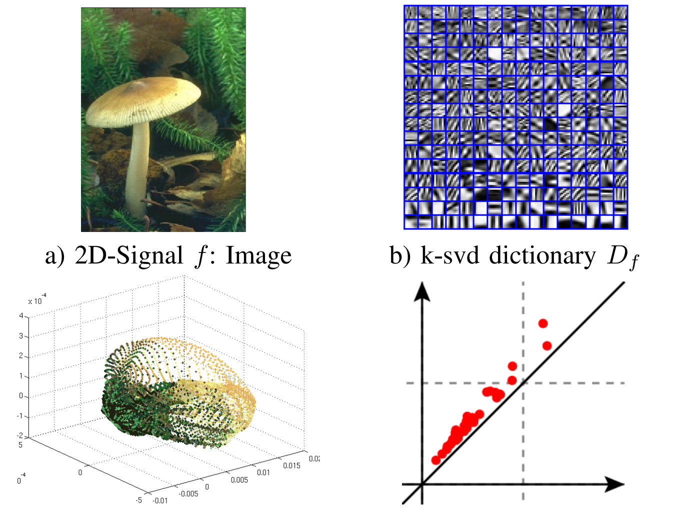
■

This proposition is an initial step towards the integration of frame theory and persistent stability. We remark that important developments have been achieved in generalizing the work in [4] and the Inequality 2 by avoiding the restrictions imposed by the functional setting and expressing the stability in a purely algebraic language (see [2]). The usage of these more flexible and general stability properties is a natural step in our program.

A. Experiments

A parallel work we undertake is to investigate with computer experiments the interaction between the components in our framework (dictionary learning, frame constructions, and persistent diagrams). A main objective is to gather further empirical understanding of the persistent diagrams stability when a dynamical process is applied to a signal. We experiment with examples including 1D and 2D signals. We consider in particular two signals f_0 and f_1, together with a process transforming f_0 into f_1 encoded with a family of signals $\{f_t\}_{0 \leq t \leq 1}$. For 2D signals, we study the set of $n \times n$ patches $X_{f_t} = X_t = \{x_i^t\}_{i \in I}$ and the corresponding k-svd dictionary construction $D_{f_t} = D_t = \{d_j^t\}_{j \in J}$ (see [1]). The fundamental objective of our experiments is to understand the interaction between the persistent diagrams of X_t and D_t. We study these interactions as a way to better design dictionaries and characterize signals for classification purposes.

The basic inputs for our experiments are 1D and 2D signals together with the corresponding k-svd dictionary construction, time-frequency transforms and persistent diagrams algorithms. In Fig. 1 we consider an image signal, together with a k-svd dictionary constructed our of the set of 8x8 patches X_f and the corresponding persistent diagram of X_f.

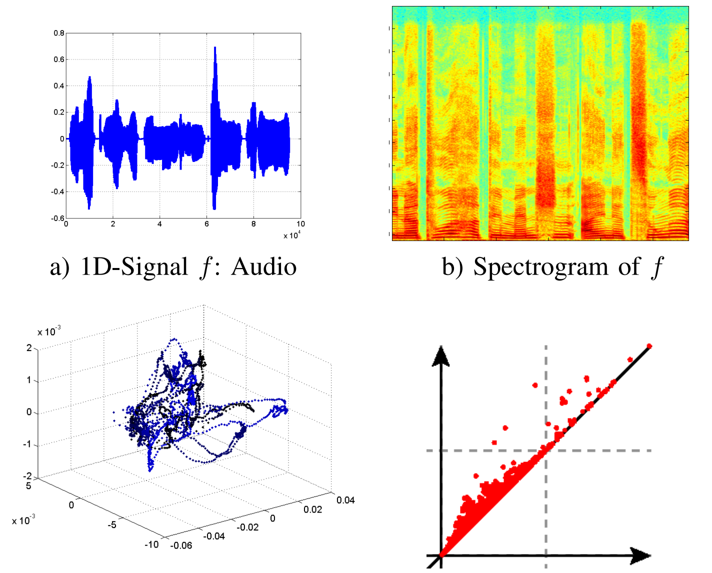


a) 2D-Signal f: Image b) k-svd dictionary D_f

c) X_f: patch dataset in $\mathbb{R}^3$ d) Persistent Diagram of X_f.

Fig. 1. 2D signals, dictionaries, patches dataset, and their persistent diagrams.

In the 1D situation of Fig. 2, we consider an acoustic signal, its time-frequency representation, the corresponding set of signal chunks X_f, and their persistent diagrams. We remark that recent algorithms proposals are combining these inputs with dictionary learning based on k-svd and non-negative matrix factorization [8].



a) 1D-Signal f: Audio b) Spectrogram of f

c) X_f: spectrogram dataset in $\mathbb{R}^3$ d) Persistent Diagram of X_f.

Fig. 2. 1D signals, time-frequency plot, its dataset, and persistent diagram.

Acknowledgment

This work is supported by the DFG Research Center Matheon Project B26: Information Extracting Sensor Networks. G. Kutyniok acknowledges support by the Einstein Foundation Berlin, by Deutsche Forschungsgemeinschaft (DFG) Grant SPP- 1324 KU 1446/13 and DFG Grant KU 1446/14, by the DFG Collaborative Research Center TRR 109 “Discretization in Geometry and Dynamics”, and by the DFG Research Center Matheon “Mathematics for key technologies” in Berlin, Germany.

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