Scalable frames and convex geometry

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Abstract. The recently introduced and characterized scalable frames can be considered as those frames which allow for perfect preconditioning in the sense that the frame vectors can be rescaled to yield a tight frame. In this paper we define $m-$scalability, a refinement of scalability based on the number of non-zero weights used in the rescaling process. We investigate the connection between this notion and elements from convex geometry. Finally, we provide some partial results on the topology of scalable frames. In particular, we prove that the set of scalable frames with "usual" redundancy is nowhere dense in the set of frames.

1. Introduction

Frame theory, both in the finite and infinite dimensional setting, has become a standard tool in mathematical signal processing and engineering. The main reason for this and, simultaneously, the key feature of frames is their redundancy which ensures their robustness against perturbations such as noise or erasures [**Dau, CK03**]. For a detailed treatment of frames in theory and applications we refer the interested reader to [**CaKu, ChrO, KoChe1, KoChe2**].

In this paper we consider frames for finite-dimensional real Euclidean spaces $\mathbb{R}^N$. In this context, a frame is a set $\Phi = \{\varphi_k\}_{k=1}^M \subset \mathbb{R}^N$, $M \geq N$, for which there exist positive constants A and B such that

$$A\|x\|^2 \;\leq\; \sum_{k=1}^{M} |\langle x, \varphi_k\rangle|^2 \;\leq\; B\|x\|^2 \tag{1.1}$$

holds for all $x \in \mathbb{R}^N$. Constants A and B as in (1.1) are called *frame bounds* of Φ. The frame Φ is called *tight* if $A = B$ is possible in (1.1). In this case we have $A = \frac{1}{N}\sum_{=1}^{M}\|\varphi_k\|^2$. A tight frame with $A = B = 1$ in (1.1) is called *Parseval frame* (see, e.g., [**BF, CFKLT**]).

We will sometimes identify a frame $\Phi = \{\varphi_k\}_{k=1}^M \subset \mathbb{R}^N$ with the $N \times M$ matrix whose kth column is the vector φ_k. This matrix is called the *synthesis operator* of the frame. The adjoint Φ^T of Φ is called the *analysis operator*. Using the analysis operator, the relation (1.1) reads

$$A\|x\|^2 \;\leq\; \|\Phi^T x\|^2 \;\leq\; B\|x\|^2.$$

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Hence, a frame Φ is tight if and only if some multiple of Φ^T is an isometry. Formulated in the language of numerical linear algebra, a tight frame is perfectly conditioned since the condition number of its analysis operator is one. It is therefore obvious that tight frames are highly advantageous over non-tight frames when applied to real world problems. It is thus desirable to construct tight frames by modifying frames in a very simple manner.

Recently, together with E.K. Tuley the authors introduced in [**KOPT12**] the so-called *scalable frames* (see also [**KOP1**]). Hereby, a frame $\Phi = \{\varphi_k\}_{k=1}^M$ is *scalable* if the frame vectors φ_k, $k = 1, \ldots, M$, can be rescaled such that the resulting frame is tight. Since the analysis operator after rescaling has the form $D\Phi^T$ with a diagonal matrix D, a scalable frame is a frame whose analysis operator allows for perfect pre-conditioning (see [**BB, C05, Kol**]), meaning that $D\Phi^T$ has condition number one. One of the main results in [**KOPT12, KOP1**] is a simple geometric characterization of the complement of the set of scalable frames in the set of all frames with a fixed number of frame vectors.

After the appearance of [**KOPT12**], scalable frames have also been investigated in the papers [**CKLMNS**] and [**CC**]. In [**CKLMNS**] the authors analyzed the problem by making use of the properties of so-called diagram vectors [**HKLW**] whereas [**CC**] gives a detailed insight into the set of weights which can be used for scaling.

In the present paper we refine the definition of scalability by calling a (scalable) frame $m-$scalable if at most m non-zero weights can be used for the scaling. Subsequently, this refinement leads to a reformulation of the scalability question in terms of the properties of certain polytopes associated to a nonlinear transformation of the frame vectors. This nonlinear transformation is related but not equivalent to the diagram vectors used in the results obtained in [**CKLMNS**]. Using this reformulation, we establish new characterizations of scalable frames, based on one of the many versions of Farkas' Lemma and further illustrate the link between scalable frames and properties of convex polytopes. Finally, we investigate the topological properties of the set of scalable frames. In particular, we prove that in the set of frames in $\mathbb{R}^N$ with M frame vectors the set of scalable frames is nowhere dense if $M < N(N+1)/2$. The results we prove here were announced in [**KOP2**].

2. Preliminaries

First of all, let us fix some notation. If X is any set whose elements are indexed x_j, $j \in J$, and $I \subset J$, we define $X_I := \{x_i : i \in I\}$. Moreover, for the set $\{1, \ldots, n\}$, $n \in \mathbb{N}$, we write $[n]$. The set of frames for $\mathbb{R}^N$ with M elements will be denoted by $\mathcal{F}(M,N)$. We say that a frame $\Phi \in \mathcal{F}(M,N)$ is *degenerate* if one of its frame vectors is the zero-vector. If $\mathcal{X}(M,N)$ is a set of frames in $\mathcal{F}(M,N)$, we denote by $\mathcal{X}^*(M,N)$ the set of the non-degenerate frames in $\mathcal{X}(M,N)$. For example, $\mathcal{F}^*(M,N)$ is the set of non-degenerate frames in $\mathcal{F}(M,N)$.

We begin by recalling the following definition from [**KOPT12**, Definition 2.1].

Definition 2.1. A frame $\Phi = \{\varphi_k\}_{k=1}^M$ for $\mathbb{R}^N$ is called *scalable*, respectively, *strictly scalable*, if there exist nonnegative, respectively, positive, scalars $c_1, \ldots, c_M \in \mathbb{R}$ such that $\{c_k\varphi_k\}_{k=1}^M$ is a tight frame for $\mathbb{R}^N$. The set of scalable, respectively, strictly scalable, frames in $\mathcal{F}(M,N)$ is denoted by $\mathcal{SC}(M,N)$, respectively, $\mathcal{SC}_+(M,N)$.

We now refine this definition in order to gain a better understanding of the structure of scalable frames.

DEFINITION 2.2. Let $M, N, m \in \mathbb{N}$ be given such that $N \leq m \leq M$. A frame $\Phi = \{\varphi_k\}_{k=1}^M \in \mathcal{F}(M,N)$ is said to be *$m-$scalable*, respectively, *strictly $m-$scalable*, if there exists a subset $I \subseteq [M]$, $\#I = m$, such that Φ_I is a scalable frame, respectively, a strictly scalable frame for $\mathbb{R}^N$. We denote the set of $m-$scalable frames, respectively, strictly $m-$scalable frames in $\mathcal{F}(M,N)$ by $\mathcal{SC}(M,N,m)$, respectively, $\mathcal{SC}_+(M,N,m)$.

It is easily seen that for $m \leq m'$ we have that $\mathcal{SC}(M,N,m) \subset \mathcal{SC}(M,N,m')$. Therefore,

$$\mathcal{SC}(M,N) = \mathcal{SC}(M,N,M) = \bigcup_{m=N}^{M} \mathcal{SC}(M,N,m).$$

We often only write $\mathcal{F}$, $\mathcal{SC}$, $\mathcal{SC}_+$, $\mathcal{SC}(m)$, and $\mathcal{SC}_+(m)$ instead of $\mathcal{SC}(M,N)$, $\mathcal{SC}_+(M,N)$, $\mathcal{SC}(M,N,m)$, and $\mathcal{SC}_+(M,N,m)$, respectively. The notations $\mathcal{F}^*$, $\mathcal{SC}^*$, $\mathcal{SC}_+^*$, $\mathcal{SC}(m)^*$, and $\mathcal{SC}_+(m)^*$ are to be read analogously.

Note that for a frame $\Phi \in \mathcal{F}$ to be m-scalable it is necessary that $m \geq N$, and given $M \geq N \geq 2$, if $\Phi = \{\varphi_k\}_{k=1}^M \in \mathcal{SC}(M,N)$, then $\Phi \in \mathcal{SC}(m)$ for some $N \leq m \leq M$. In addition, $\Phi \in \mathcal{SC}(M,N)$ holds if and only if $T(\Phi) \in \mathcal{SC}(M,N)$ for one (and hence for all) orthogonal transformation(s) T on $\mathbb{R}^N$; cf. [**KOPT12**, Corollary 2.6].

If $M \geq N$, we have $\Phi \in \mathcal{SC}(M,N,N)$ if and only if Φ contains an orthogonal basis of $\mathbb{R}^N$. This completely characterizes the set of $N-$scalable frames of $M \geq N$ vectors in $\mathbb{R}^N$. For frames with $M = N+1$ vectors in $\mathbb{R}^N$ we have the following result:

PROPOSITION 2.3. *Let $N \geq 2$ and $\Phi = \{\varphi_k\}_{k=1}^{N+1} \in \mathcal{F}^*$ with $\varphi_k \neq \pm\varphi_\ell$ for $k \neq \ell$. If $\Phi \in \mathcal{SC}_+(N+1,N,N)$ then $\Phi \notin \mathcal{SC}_+(N+1,N,N+1)$. In particular,*

$$\mathcal{SC}_+(N+1,N,N+1) \cap \mathcal{SC}_+(N+1,N,N) = \varnothing.$$

PROOF. If $\Phi \in \mathcal{SC}_+(N+1,N,N)$, then Φ must contain an orthogonal basis. By applying some orthogonal transformation and rescaling the frame vectors, we can assume without loss of generality that $\{\varphi_k\}_{k=1}^N = \{e_k\}_{k=1}^N$ is the standard orthonormal basis of $\mathbb{R}^N$, and that $\varphi_{N+1} \neq \pm e_k$ for each $k = 1,2,\dots,N$, with $\|\varphi_{N+1}\| = 1$. Thus, Φ can be written as $\Phi = \begin{bmatrix}\mathrm{Id}_N & \varphi_{N+1}\end{bmatrix}$, where Id_N is the $N \times N$ identity matrix.

Assume that there exists $\{\lambda_k\}_{k=1}^{N+1} \subset (0,\infty)$ such that $\widetilde{\Phi} = \{\lambda_k\varphi_k\}_{k=1}^{N+1}$ is a tight frame, i.e. $\widetilde{\Phi}\widetilde{\Phi}^T = A\,\mathrm{Id}_N$.

Using a block multiplication this equation can be rewritten as

$$\Lambda + \lambda_{N+1}^2 \varphi_{N+1}\varphi_{N+1}^T = A\,\mathrm{Id}_N$$

where $\Lambda = \mathrm{diag}(\lambda_k^2)$ is the $N \times N$ diagonal matrix with λ_k^2, $k = 1,\dots,N$, on its diagonal. Consequently,

$$\lambda_k^2 + \lambda_{N+1}^2\varphi_{N+1}(k) = A \text{ for } k = 1,\dots,N \quad \text{and} \quad \lambda_{N+1}^2\varphi_{N+1}(\ell)\varphi_{N+1}(k) = 0 \text{ for } k \neq \ell.$$

But $\lambda_{N+1} > 0$ and so all but one entry in φ_{N+1} vanish. Since φ_{N+1} is a unit norm vector, we see that $\varphi_{N+1} = \pm e_k$ for some $k = 1,\dots,N$ which is contrary to the assumption, so Φ cannot be strictly $(N+1)-$scalable. Thus, $\mathcal{SC}_+(N+1,N,N+1) \cap \mathcal{SC}_+(N+1,N,N) = \varnothing$. $\square$

We conclude this section with the following remark:

REMARK 2.4. To determine that a frame Φ is scalable, we could always assume without loss of generality that all the frame vectors are in the upper-half plane, that is $\Phi \subset \mathbb{R}^{N-1} \times \mathbb{R}_{+,0}$ where $\mathbb{R}_{+,0} = [0, \infty)$. Indeed, given a frame $\Phi \subset \mathbb{R}^N$, assume that $\Phi = \Phi_1 \cup \Phi_2$ where

$$\Phi_1 = \{\varphi_k^{(1)} \in \Phi : \varphi_k^{(1)}(N) \geq 0\}$$

and

$$\Phi_2 = \{\varphi_k^{(2)} \in \Phi : \varphi_k^{(2)}(N) < 0\}.$$

Then the frame $\Phi' = \Phi_1 \cup (-\Phi_2) = \{\varphi_k^{(1)}\} \cup \{-\varphi_k^{(2)}\}$ has the same frame operator as Φ. In particular, if one is a tight frame so is the other. In addition, Φ is scalable if and only if Φ' is scalable with exactly the same set of weights.

3. Scalable Frames and Convex Polytopes

Our characterizations of $m-$scalable frames will be stated in terms of certain convex polytopes and more generally using tools from convex geometry. Therefore, we collect below some key facts and properties needed to state and prove our results, and we refer to [**Mat, StWi, Web**] for details.

3.1. Background on Convex Geometry. Let $X = \{x_i\}_{i=1}^M$ be a finite set in a real linear space E. The *convex hull generated by* X is the compact convex subset of E defined by

$$\operatorname{co}(X) := \left\{ \sum_{i=1}^M \alpha_i x_i : \alpha_i \geq 0,\, \sum_{i=1}^M \alpha_i = 1 \right\}.$$

The *affine hull generated by* X is defined by

$$\operatorname{aff}(X) := \left\{ \sum_{i=1}^M \alpha_i x_i : \sum_{i=1}^M \alpha_i = 1 \right\}.$$

Hence, we have $\operatorname{co}(X) \subset \operatorname{aff}(X)$. Recall that for fixed $a \in \operatorname{aff}(X)$, the set

$$V(X) := \operatorname{aff}(X) - a = \{y - a : y \in \operatorname{aff}(X)\}$$

is a subspace of E (which is independent of $a \in \operatorname{aff}(X)$) and that one defines

$$\dim X := \dim \operatorname{co}(X) := \dim \operatorname{aff}(X) := \dim V(X).$$

We shall use Carathéodory's Theorem for convex polytopes [**StWi**, Theorem 2.2.12], in deciding whether a frame is scalable:

THEOREM 3.1 (Carathéodory). *Let $X = \{x_1, \ldots, x_k\}$ be a finite subset of E with $d := \dim X$. Then for each $x \in \operatorname{co}(X)$ there exists $I \subset [k]$ with $\#I = d + 1$ such that $x \in \operatorname{co}(X_I)$.*

The *relative interior* of the polytope $\operatorname{co}(X)$ denoted by $ri \operatorname{co}(X)$, is the interior of $\operatorname{co}(X)$ in the topology induced by $\operatorname{aff}(X)$. It is true that $ri \operatorname{co}(X) \neq \varnothing$ as long as $\#X \geq 2$; cf. [**StWi**, Lemma 3.2.8]. Furthermore,

$$ri \operatorname{co}(X) := \left\{ \sum_{i=1}^M \lambda_i x_i : \lambda_i > 0, \sum_{i=1}^M \lambda_i = 1 \right\},$$

see [**Web**, Theorem 2.3.7]. Moreover, the interior of $\operatorname{co}(X)$ in E is non-empty if and only if $\operatorname{aff}(X) = E$.

The following lemma characterizes $\dim X$ in terms of $\dim \operatorname{span} X$.

LEMMA 3.2. *Let X be a finite set of points in E. Put $m := \dim \operatorname{span} X$. Then $\dim X \in \{m-1, m\}$. Moreover, the following statements are equivalent:*

(i) $\dim X = m - 1$.
(ii) *For all linearly independent $X' \subset X$ with $\dim \operatorname{span} X' = m$ we have $X \setminus X' \subset \operatorname{aff}(X')$.*
(iii) *For some linearly independent $X' \subset X$ with $\dim \operatorname{span} X' = m$ we have $X \setminus X' \subset \operatorname{aff}(X')$.*

PROOF. Let $X = \{x_1, \dots, x_k\}$. First of all, we observe that for a linearly independent set $X' = \{x_{i_1}, \dots, x_{i_m}\}$ as in (ii) or (iii) we have

$$\dim V(X') = \dim \operatorname{span}\{x_{i_l} - x_{i_1} : l = 2, \dots, m\} = m - 1.$$

Therefore, $V(X') \subset V(X) \subset \operatorname{span} X$ implies $m - 1 \le \dim X \le m$. Let us now prove the moreover-part of the lemma.

(i)$\Rightarrow$(ii). Assume that $\dim X = m-1$ and let $X' = \{x_{i_1}, \dots, x_{i_m}\}$ be a linearly independent set as in (ii). From $\dim V(X) = \dim X = m - 1$ we obtain $V(X) = V(X')$. Therefore, for each $x_j \in X \setminus X'$ there exist $\mu_2, \dots, \mu_m \in \mathbb{R}$ such that

$$x_j - x_{i_1} = \sum_{i=2}^{m} \mu_i (x_i - x_{i_1}) = \sum_{i=2}^{m} \mu_i x_i - \left(\sum_{i=2}^{m} \mu_i \right) x_{i_1}.$$

And this implies

$$x_j = \left(1 - \sum_{i=2}^{m} \mu_i \right) x_{i_1} + \sum_{i=2}^{m} \mu_i x_i \in \operatorname{aff}(X').$$

(ii)$\Rightarrow$(iii). This is obvious.

(iii)$\Rightarrow$(i). Let $X' = \{x_{i_1}, \dots, x_{i_m}\}$ be a linearly independent set as in (iii). If $x \in X \setminus X'$, then we have $x \in \operatorname{aff}(X')$ by (iii). Consequently, there exist $\lambda_1, \dots, \lambda_m \in \mathbb{R}$ with $\sum_{l=1}^m \lambda_l = 1$ such that $x = \sum_{l=1}^m \lambda_l x_{i_l}$. Hence, we obtain

$$x - x_{i_1} = \sum_{l=1}^{m} \lambda_l x_{i_l} - \left(\sum_{l=1}^{m} \lambda_l \right) x_{i_1} = \sum_{l=1}^{m} \lambda_l (x_{i_l} - x_{i_1}) \in V(X').$$

This implies $V(X) = V(X')$ and hence (i). $\square$

In the sequel we shall use a special case of Lemma 3.2, where X is a set of rank-one orthogonal projections. More specifically,

COROLLARY 3.3. *If the set X in Lemma 3.2 consists of rank-one orthogonal projections acting on a real Hilbert space, then we have*

$$\dim X = \dim \operatorname{span} X - 1.$$

PROOF. Let $X = \{P_1, \dots, P_k\}$, $m := \dim \operatorname{span} X$, and let $X' \subset X$ be a linearly independent subset of X such that $\dim \operatorname{span} X' = m$. Without loss of generality assume that $X' = \{P_1, \dots, P_m\}$. Let $j \in \{m+1, \dots, k\}$. Then there exist $\lambda_1, \dots, \lambda_m \in \mathbb{R}$ such that $P_j = \sum_{i=1}^m \lambda_i P_i$. This implies

$$1 = \operatorname{Tr} P_j = \operatorname{Tr} \left(\sum_{i=1}^{m} \lambda_i P_i \right) = \sum_{i=1}^{m} \lambda_i \operatorname{Tr}(P_i) = \sum_{i=1}^{m} \lambda_i,$$

which shows that $P_j \in \operatorname{aff}(X')$. The statement now follows from Lemma 3.2. $\square$

3.2. Scalability in Terms of Convex Combinations of Rank-One Matrices. For a frame $\Phi = \{\varphi_i\}_{i=1}^M$ in $\mathcal{F}(M,N)$ we set

$$X_\Phi := \{\varphi_i\varphi_i^T : i \in [M]\}.$$

This is a subset of S_N, the space of all real symmetric $N \times N$-matrices. We shall also denote the set of positive multiples of the identity by $\boldsymbol{I}_+ := \{t\,\mathrm{Id}_N : t > 0\}$.

PROPOSITION 3.4. *For a frame $\Phi \in \mathcal{F}(M,N)$ the following statements are equivalent:*

(i) *Φ is scalable, respectively, strictly scalable.*
(ii) *$\boldsymbol{I}_+ \cap \operatorname{co}(X_\Phi) \neq \varnothing$, respectively, $\boldsymbol{I}_+ \cap ri\operatorname{co}(X_\Phi) \neq \varnothing$.*

PROOF. Assume that the frame $\Phi = \{\varphi_i\}_{i=1}^M$ is scalable. Then there exist non-negative scalars $c_1, \ldots, c_M$ such that

$$\sum_{i=1}^M c_i\varphi_i\varphi_i^T = \mathrm{Id}\,.$$

Put $\alpha := \sum_{i=1}^M c_i$. Then $\alpha > 0$ and with $\lambda_i := \alpha^{-1}c_i$ we have

$$\sum_{i=1}^M \lambda_i\varphi_i\varphi_i^T = \alpha^{-1}\,\mathrm{Id} \quad \text{and} \quad \sum_{i=1}^M \lambda_i = 1.$$

Hence $\alpha^{-1}\,\mathrm{Id} \in \operatorname{co}(X_\Phi)$. The converse direction is obvious. $\square$

As pointed out earlier, for $m \leq m'$, $\mathcal{SC}(m) \subset \mathcal{SC}(m')$. The next result shows that given $\Phi \in \mathcal{SC}(M,N)$, there is a generic $N \leq m := m_\Phi \leq M$ such that $\Phi \in \mathcal{SC}(m)$.

PROPOSITION 3.5. *A frame $\Phi = \{\varphi_k\}_{k=1}^M \in \mathcal{F}$ is scalable if and only if it is m-scalable, where $m := \dim\operatorname{span} X_\Phi$.*

PROOF. Clearly, an m-scalable frame is scalable. Conversely, let $\Phi = \{\varphi_i\}_{i=1}^M$ be scalable. After possibly removing zero vectors from the frame and thereby reducing M (which does not affect the value of m), we may assume that Φ is unit-norm. By Proposition 3.4, there exists $\alpha > 0$ such that $\alpha\,\mathrm{Id} \in \operatorname{co}(X_\Phi)$.

Therefore, from Theorem 3.1 it follows that there exists $I \subset [M]$ with $\#I = \dim X_\Phi + 1$ such that $\alpha\,\mathrm{Id} \in \operatorname{co}(X_I)$. Hence, Φ_I is scalable by Proposition 3.4. And since $\dim X_\Phi = \dim\operatorname{span} X_\Phi - 1$ by Corollary 3.3, the claim follows. $\square$

As $X_\Phi \subset S_N$ and $\dim S_N = N(N+1)/2$, we immediately obtain the following corollary.

COROLLARY 3.6. *For $M \geq N(N+1)/2$ we have*

$$\mathcal{SC}(M,N) = \mathcal{SC}\left(M, N, \frac{N(N+1)}{2}\right).$$

3.3. Convex Polytopes Associated with m−Scalable Frames. The $m-$scalability of a frame $\Phi = \{\varphi_k\}_{k=1}^M$ is equivalent to the existence of nonnegative numbers $\{c_k\}_{k\in I}$ with $\#I = m \geq N$ such that

$$\Phi C^2 \Phi^T = A\,\mathrm{Id}_N, \tag{3.1}$$

where $A > 0$ and C is the diagonal matrix with the weights c_k on its diagonal for $k \in I$ and 0 on for $k \notin I$. Note that this automatically implies $N \leq \#\{k \in I : c_k > 0\} \leq m \leq M$. Comparing corresponding entries from left- and right-hand sides of (3.1), it is seen that for a frame to be $m-$scalable it is necessary and sufficient that there exists a vector $u = (c_1^2, c_2^2, \ldots, c_M^2)^T$ with $\|u\|_0 := \#\{k \in [M] : c_k \neq 0\} \leq m$ which is a solution of the following linear system of $\frac{N(N+1)}{2}$ equations in M unknowns:

$$\begin{cases} \sum\limits_{j=1}^{M} \varphi_j(k)^2 y_j = A & \text{for } k = 1, \ldots, N, \\ \sum\limits_{j=1}^{M} \varphi_j(\ell)\varphi_j(k) y_j = 0 & \text{for } \ell, k = 1, \ldots, N,\ k > \ell. \end{cases} \tag{3.2}$$

Subtraction of equations in the first system in (3.2) leads to the new *homogeneous* linear system

$$\begin{cases} \sum\limits_{j=1}^{M} \big(\varphi_1(1)^2 - \varphi_j(k)^2\big) y_j = 0 & \text{for } k = 2, \ldots, N, \\ \sum\limits_{j=1}^{M} \varphi_j(\ell)\varphi_j(k) y_j = 0 & \text{for } \ell, k = 1, \ldots, N,\ k > \ell. \end{cases} \tag{3.3}$$

It is not hard to see that we have not lost information in the last step, hence Φ is $m-$scalable if and only if there exists a nonnegative vector $u \in \mathbb{R}^M$ with $\|u\|_0 \leq m$ which is a solution to (3.3). In matrix form, (3.3) reads

$$F(\Phi)u = 0,$$

where the $(N-1)(N+2)/2 \times M$ matrix $F(\Phi)$ is given by

$$F(\Phi) = \begin{pmatrix} F(\varphi_1) & F(\varphi_2) & \ldots & F(\varphi_M) \end{pmatrix},$$

where $F : \mathbb{R}^N \to \mathbb{R}^d$, $d := (N-1)(N+2)/2$, is defined by

$$F(x) = \begin{pmatrix} F_0(x) \\ F_1(x) \\ \vdots \\ F_{N-1}(x) \end{pmatrix}, \qquad F_0(x) = \begin{pmatrix} x_1^2 - x_2^2 \\ x_1^2 - x_3^2 \\ \vdots \\ x_1^2 - x_N^2 \end{pmatrix}, \qquad F_k(x) = \begin{pmatrix} x_k x_{k+1} \\ x_k x_{k+2} \\ \vdots \\ x_k x_N \end{pmatrix},$$

and $F_0(x) \in \mathbb{R}^{N-1}$, $F_k(x) \in \mathbb{R}^{N-k}$, $k = 1, 2, \ldots, N-1$.

Summarizing, we have just proved the following proposition.

Proposition 3.7. *A frame Φ for $\mathbb{R}^N$ is (strictly) $m-$scalable if and only if there exists a nonnegative $u \in \ker F(\Phi) \setminus \{0\}$ with $\|u\|_0 \leq m$ (respectively, $\|u\|_0 = m$).*

We will now utilize the above reformulation to characterize $m-$scalable frames in terms of the properties of convex polytopes of the type $\mathrm{co}(F(\Phi_I))$, $I \subset [M]$. One of the key tools will be the Farkas lemma (see, e.g., [**Mat**, Lemma 1.2.5]).

Lemma 3.8. *For every real $N \times M$-matrix A exactly one of the following cases occurs:*

(i) *The system of linear equations* $Ax = 0$ *has a nontrivial nonnegative solution* $x \in \mathbb{R}^M$ (*all components of* x *are nonnegative and at least one of them is strictly positive.*)
(ii) *There exists* $y \in \mathbb{R}^N$ *such that* $y^T A$ *is a vector with all entries strictly positive.*

Our first main result can now be stated as follows:

THEOREM 3.9. *Let* $M \geq N \geq 2$, *and let* m *be such that* $N \leq m \leq M$. *Assume that* $\Phi = \{\varphi_k\}_{k=1}^M \in \mathcal{F}^*(M,N)$ *is such that* $\varphi_k \neq \varphi_\ell$ *when* $k \neq \ell$. *Then the following statements are equivalent:*

(i) Φ *is* $m-$*scalable, respectively, strictlym*$-$*scalable,*
(ii) *There exists a subset* $I \subset [M]$ *with* $\#I = m$ *such that* $0 \in \operatorname{co}(F(\Phi_I))$, *respectively,* $0 \in ri\operatorname{co}(F(\Phi_I))$.
(iii) *There exists a subset* $I \subset [M]$ *with* $\#I = m$ *for which there is no* $h \in \mathbb{R}^d$ *with* $\langle F(\varphi_k), h\rangle > 0$ *for all* $k \in I$, *respectively, with* $\langle F(\varphi_k), h\rangle \geq 0$ *for all* $k \in I$, *with at least one of the inequalities being strict.*

PROOF. (i)$\Leftrightarrow$(ii). This equivalence follows directly if we can show the following equivalences for $\Psi \subset \Phi$:

$$\begin{aligned} 0 \in \operatorname{co}(F(\Psi)) &\iff \ker F(\Psi) \setminus \{0\} \text{ contains a nonnegative vector and} \\ 0 \in ri\operatorname{co}(F(\Psi)) &\iff \ker F(\Psi) \text{ contains a positive vector.} \end{aligned} \tag{3.4}$$

The implication "$\Rightarrow$" is trivial in both cases. For the implication "$\Leftarrow$" in the first case let $I \subset [M]$ be such that $\Psi = \Phi_I$, $I = \{i_1, \dots, i_m\}$, and let $u = (c_1, \dots, c_m)^T \in \ker F(\Psi)$ be a non-zero nonnegative vector. Then $A := \sum_{k=1}^m c_k > 0$ and with $\lambda_k := c_k/A$, $k \in [m]$, we have $\sum_{k=1}^m \lambda_k = 1$ and $\sum_{k=1}^m \lambda_k F(\varphi_{i_k}) = A^{-1}F(\Psi)u = 0$. Hence $0 \in \operatorname{co}(F(\Psi))$. The proof for the second case is similar.

(ii)$\Leftrightarrow$(iii). In the non-strict case this follows from (3.4) and Lemma 3.8 In the strict case this is a known fact; e.g., see [**Web**, Lemma 3.6.5]. $\square$

We now derive a few consequences of the above theorem. A given a vector $v \in \mathbb{R}^d$ defines a hyperplane by

$$H(v) = \{y \in \mathbb{R}^d : \langle v, y\rangle = 0\},$$

which itself determines two open convex cones $H^-(v)$ and $H^+(v)$, defined by

$$H^-(v) = \{y \in \mathbb{R}^d : \langle v, y\rangle < 0\} \quad \text{and} \quad H^+(v) = \{y \in \mathbb{R}^d : \langle v, y\rangle > 0\}.$$

Using these notations we can restate the equivalence (i)$\Leftrightarrow$(iii) in Theorem 3.9 as follows:

PROPOSITION 3.10. *Let* $M \geq N \geq 2$, *and let* m *be such that* $N \leq m \leq M$. *Assume that* $\Phi = \{\varphi_k\}_{k=1}^M \in \mathcal{F}^*$ *is such that* $\varphi_k \neq \varphi_\ell$ *when* $k \neq \ell$. Φ *is* $m-$*scalable if and only if there exists a subset* $I \subset [M]$ *with* $\#I = m$ *such that* $\bigcap_{i \in I} H^+(F(\varphi_i)) = \varnothing$.

REMARK 3.11. In the case of strict $m-$scalability we have the following necessary condition. If Φ is strictly $m-$scalable, then there exists a subset $I \subset [M]$ with $\#I = m$ such that $\bigcap_{i \in I} H^-(F(\varphi_i)) = \varnothing$.

Moreover assume that $M \geq N \geq 2$, and let m be such that $N \leq m \leq M$. Suppose that $\Phi = \{\varphi_k\}_{k=1}^M \in \mathcal{F}^*$ is such that $\varphi_k \neq \varphi_\ell$ when $k \neq \ell$. If for each subset $I \subset [M]$ with $\#I = m$ we have $\bigcap_{i \in I} H^+(F(\varphi_i)) \neq \varnothing$ (or $\bigcap_{i \in I} H^-(F(\varphi_i)) \neq \varnothing$), then Φ is not $m-$scalable.

REMARK 3.12. When $M \geq d+1$, we can use properties of the convex sets $H^{\pm}(F(\varphi_k))$ to give an alternate proof of Corollary 3.6. In particular, assume that $m \geq d+1 = N(N+1)/2$. If $\Phi = \{\varphi_k\}_{k=1}^M \in \mathcal{F}^*$ with $\varphi_k \neq \varphi_\ell$ when $k \neq \ell$ is $m-$scalable then Φ is $(d+1)-$scalable. Consequently , for $m \geq d+1$, $\mathcal{SC}(m) = \mathcal{SC}(d+1)$, and $\mathcal{SC}(M,N) = \mathcal{SC}(M,N,d+1)$.

To see that this is true, observe that when $m = d+1$ there is nothing to prove. Thus, we can assume that Φ is $m-$scalable for some $N \leq m \leq M$ and that $m \geq d+2$. Then, there exists $I \subset \{1,2,\ldots,M\}$ with $\#I = m \geq d+2$ such that $\bigcap_{k\in I} H^+(F(\varphi_k)) = \varnothing$. We claim that there exists $I_0 \subset I$ with $\#I_0 = d+1$ such that $\bigcap_{i\in I_0} H^+(F(\varphi_i)) = \varnothing$. Indeed, if this was not the case, we will have for every subset $I' \subset I$ with $\#I' = d+1$, $\bigcap_{k\in I'} H^+(F(\varphi_k)) \neq \varnothing$. Because each $i \in I$, $H^+(F(\varphi_i)) \subset \mathbb{R}^d$ is convex, we can use Helly's theorem [**Mat**, Theorem 1.3.2], to conclude that $\bigcap_{i\in I} H^+(F(\varphi_i)) \neq \varnothing$. Which contradicts the choice of I.

Consequently, there exists $I_0 \subset I$ with $\#I_0 = d+1$ such that $\bigcap_{i\in I_0} H^+(F(\varphi_i)) = \varnothing$. Hence, using Lemma A and Proposition 3.10, we conclude that Φ is $(d+1)-$scalable.

The following result is an application of Proposition 3.10 which provides an easy condition for $\Phi \notin \mathcal{SC}(M,N)$. It relies on the fact that $\Phi \in \mathcal{SC}(M,N)$ if and only if $T(\Phi) \in \mathcal{SC}(M,N)$ for all orthogonal transformations T on $\mathbb{R}^N$, [**KOPT12**, Corollary 2.6].

PROPOSITION 3.13. *Let $\Phi = \{\varphi_k\}_{k=1}^M$ be a frame for $\mathbb{R}^N$, $N \geq 2$. If there exists an isometry T such that $T(\Phi) \subset \mathbb{R}^{N-2} \times \mathbb{R}^2_+$, then Φ is not scalable.*

PROOF. Without any loss of generality assume that $\Phi \subset \mathbb{R}^{N-2} \times \mathbb{R}^2_+$, and let $\{e_k\}_{k=1}^d$ be the standard ONB for $\mathbb{R}^d$. Then for each $k \in [M]$ we have that

$$\langle e_d, F(\varphi_k)\rangle = \varphi_k(N-1)\varphi_k(N) > 0.$$

Hence, $e_d \in \bigcap_{i\in[M]} H^+(F(\varphi_i))$. By Proposition 3.10, Φ is not scalable. □

The characterizations stated above can be recast in terms of the convex cone $C(F(\Phi))$ generated by $F(\Phi)$. We state this result for the sake of completeness. But first, recall that for a finite subset $X = \{x_1,\ldots,x_M\}$ of $\mathbb{R}^d$ the *polyhedral cone generated by* X is the closed convex cone set $C(X)$ defined by

$$C(X) = \left\{\sum_{i=1}^M \alpha_i x_i : \alpha_i \geq 0\right\}.$$

Let C be a cone in $\mathbb{R}^d$. The *polar cone* of C is the closed convex cone C° defined by

$$C^\circ := \{x \in \mathbb{R}^N : \langle x, y\rangle \leq 0 \text{ for all } y \in C\}.$$

The cone C is said to be *pointed* if $C \cap (-C) = \{0\}$, and *blunt* if the linear space generated by C is $\mathbb{R}^N$, i.e. $\operatorname{span} C = \mathbb{R}^N$.

COROLLARY 3.14. *Let $\Phi = \{\varphi_k\}_{k=1}^M \in \mathcal{F}^*$, and let $N \leq m \leq M$ be fixed. Then the following conditions are equivalent:*

(i) *Φ is strictly $m-$scalable .*
(ii) *There exists $I \subset [M]$ with $\#I = m$ such that $C(F(\Phi_I))$ is not pointed.*
(iii) *There exists $I \subset [M]$ with $\#I = m$ such that $C(F(\Phi_I))^\circ$ is not blunt.*

(iv) *There exists* $I \subset [M]$ *with* $\#I = m$ *such that the interior of* $C(F(\Phi_I))^\circ$ *is empty.*

Proof. (i)⇔(ii). By Proposition 3.7, Φ is strictly $m-$scalable if and only if there exist $I \subset [M]$ with $\#I = m$ and a nonnegative $u \in \ker F(\Phi_I) \setminus \{0\}$ with $\|u\|_0 = m$. By [**StWi**, Lemma 2.10.9], this is equivalent to the cone $C(F(\Phi_I))$ being not pointed. This proves that (i) is equivalent to (ii).

(ii)⇔(iii). This follows from the fact that the polar of a pointed cone C is blunt and vice versa, as long as $C^{\circ\circ} = C$, see [**StWi**, Theorem 2.10.7]. But in our case $C(F(\Phi_I))^{\circ\circ} = C(F(\Phi_I))$, see [**StWi**, Lemma 2.7.9].

(iii)⇒(iv). If $C(F(\Phi_I))^\circ$ is not blunt, then it is contained in a proper hyperplane of $\mathbb{R}^d$ whose interior is empty. Hence, also the interior of $C(F(\Phi_I))^\circ$ must be empty.

(iv)⇒(iii). We use a contra positive argument. Assume that $C(F(\Phi))^\circ$ is blunt. This is equivalent to $\operatorname{span} C(F(\Phi))^\circ = \mathbb{R}^d$. But for the nonempty cone $C(F(\Phi))^\circ$ we have $\operatorname{aff}(C(F(\Phi))^\circ) = \operatorname{span} C(F(\Phi))^\circ$. Hence, $\operatorname{aff}(C(F(\Phi))^\circ) = \mathbb{R}^d$, and so the relative interior of $C(F(\Phi))^\circ$ is equal to its interior, which therefore is nonempty. □

The main ideas of the previous results is the characterization of the scalability of Φ in terms of properties of the convex polytope $\operatorname{co}(F(\Phi))$. One could ask what assumptions must the convex set $\operatorname{co}(\Phi)$ satisfy to ensure that the conditions in Theorem 3.9 and some of its corollaries hold. This is a very interesting question that we have not address here, but that we shall investigate elsewhere. Proposition 3.13 that gives a condition on Φ that precludes it to be scalable, is a step in this direction.

Nonetheless, we address the related question of whether $F(\Phi)$ is a frame for $\mathbb{R}^d$ whenever Φ is a frame for $\mathbb{R}^N$. This depends clearly on the redundancy of Φ as well as on the map F. In particular, we finish this section by giving a condition that ensures that $F(\Phi)$ is always a frame for $\mathbb{R}^d$ when $M \geq d+1$. In order to prove this result, we need a few preliminary facts.

For $x = (x_k)_{k=1}^N \in \mathbb{R}^N$ and $h = (h_k)_{k=1}^d \in \mathbb{R}^d$, we have that

(3.5)
$$\langle F(x), h\rangle = \sum_{k=2}^{N} h_k(x_1^2 - x_k^2) + \sum_{k=1}^{N-1}\sum_{\ell=1}^{N-k} h_{kN-\frac{k(k-1)}{2}+\ell}\, x_k x_\ell =: p(x_1, x_2, \ldots, x_N).$$

p is a homogeneous polynomial of degree 2 in $x_1, x_2, \ldots, x_N$ and consists of monomials of degree 2 as well. We shall call the set of all such polynomials of the form given by (3.5) $\boldsymbol{P}_2^N$. It is easily seen that $\boldsymbol{P}_2^N$ is isomorphic to the subspace of real symmetric $N \times N$ matrices whose trace is 0.

Indeed, for each $N \geq 2$, and each $p \in \boldsymbol{P}_2^N$,

$$p(x_1, x_2, \ldots, x_N) = \sum_{k=2}^{N} a_k(x_1^2 - x_k^2) + \sum_{k=1}^{N-1}\sum_{\ell=1}^{N-k} a_{kN-\frac{k(k-1)}{2}+\ell}\, x_k x_\ell = \langle x, Q_p x\rangle$$

where $a_k \in \mathbb{R}$ for each $k = 1, 2, \ldots, (N-1)(N+2)/2$, is associated the $N \times N$ symmetric matrix Q_p

$$Q_p(1,1) = \sum_{k=1}^{N-1} a_k, \qquad Q_p(i,i) = -a_{i-1} \text{ for } i = 2, 3, \ldots, N$$

and $Q_p(i,j) = a_{i+j}/2$ for $i \neq j$. Moreover, the dimension of $\boldsymbol{P}_2^N$ is $d = (N+2)(N-1)/2$.

Using this fact and Proposition 3.5, we show that when $M \geq d+1$ (equivalently, $M/N \geq (N+1)/2$), and if $\Phi \in \mathcal{SC}_+(d+1)$, then $\mathrm{co}(F(\Phi))$ is a "full" polytope. We shall also need the following version of Farkas's lemma:

LEMMA B.([**Roc**, Corollary 22.3.1]) For every real $N \times M$-matrix A and every $b \in \mathbb{R}^N \setminus \{0\}$ the following two conditions are equivalent:

(i) The system of linear equations $Ax = b$ has a nonnegative solution $x \in \mathbb{R}^M$.
(ii) There exists no $y \in \mathbb{R}^N$ such that $y^T A \geq 0$ and $\langle y, b\rangle < 0$.

PROPOSITION 3.15. *Let $M \geq d+1$ where $d = (N-1)(N+2)/2$, and $\Phi = \{\varphi_k\}_{k=1}^M \in \mathcal{SC}_+(d+1)$ then $\dim \mathrm{co}(F(\Phi)) = d$. Consequently, $F(\Phi)$ is a frame for $\mathbb{R}^d$.*

PROOF. Since Φ is strictly $(d+1)-$scalable, there exists $\Phi_I \subset \Phi$ with $\#I = d+1$ such that for a strictly positive vector $x \in \mathbb{R}^{d+1}$ such that $F(\Phi_I)x = 0$. Hence, $0 \in \mathrm{co}(F(\Phi_I))$. We only need to prove that $\mathrm{co}(F(\Phi_I))$ contains a linearly independent set consisting of d elements.

Let $\{e_k\}_{k=1}^d$ be the standard orthonormal basis of $\mathbb{R}^d$. We claim that for each $k = 1, 2, \ldots, d$ $e_k = F(\Phi_I)x_k$ for some nonnegative vector $x_k \in \mathbb{R}^{d+1}$. We prove this using Farkas's lemma (Lemma B). Assume that there exists $y \in \mathbb{R}^d$ such that $\langle y, F(\varphi_k)\rangle \geq 0$ and $\langle y, e_k\rangle = y_k < 0$ for each $k \in I$. In particular, $y \neq 0$. Now if for at least one $k \in I$, $\langle y, F(\varphi_k)\rangle > 0$, then we would have found $y \neq 0$ and $\langle y, F(\varphi_k)\rangle \geq 0$ for all $k \in I$ with at least one strict inequality. But by Theorem 3.9 this contradicts the fact that Φ is $(d+1)-$scalable.

Thus we must have that $\langle y, F(\varphi_k)\rangle = 0$ for all $k \in I$. But this is equivalent to $\langle \varphi_k, Q_y\varphi_k\rangle = 0$ for all $k \in I$, and where $Q_y \in S^N$ is a symmetric zero trace matrix. However, this is equivalent to

$$\langle \varphi_k, Q_y \varphi_k\rangle = \mathrm{Tr}(\varphi_k \varphi_k^T Q_y) = 0$$

for all $k \in I$. But, by Proposition 3.5 $\{\varphi_k\varphi_k^T\}_{k=1}^M$ form a complete set. Hence, $Q_y = 0$ but $y \neq 0$. But this is again a contradiction. Hence, such a y cannot exists and thus for each $k \in I$, $e_k = F(\Phi_I)x_k$ for some nonnegative vector $x \in \mathbb{R}^{d+1}$. Hence, $\{e_k\}_{k=1}^d \cup \{0\}$ is an affinely independent set in $\mathrm{co}(F(\Phi_I) \subset \mathrm{co}(F(\Phi))$. $\square$

4. Topology of the Set of Scalable Frames

In this section, we present some topological features of the set $\mathcal{SC}(M,N)$. Hereby, we identify frames in $\mathcal{F}(M,N)$ with real $N \times M$ matrices as we already did before, see, e.g., (3.1) in subsection 3.3. Hence, we consider $\mathcal{F}(M,N)$ as the set of all matrices in $\mathbb{R}^{N\times M}$ of rank N. Note that under this identification the order of the vectors in a frame becomes important.

In [**KOPT12**] it was proved that $\mathcal{SC}(M,N)$ is a closed set in $\mathcal{F}(M,N)$ (in the relative topology of $\mathcal{F}(M,N)$). The next proposition refines this fact.

PROPOSITION 4.1. *Given $M \geq N \geq 2$, the following statements hold:*

(1) *If $N \leq m \leq M$, then $\mathcal{SC}(M,N,m)$ is closed in $\mathcal{F}(M,N)$.*
(2) *$\mathcal{SC}_+(N+1,N,N+1)$ is open in the relative topology generated by $\mathcal{SC}(N+1,N)$. Consequently, when $M \geq N+1$, $\mathcal{SC}_+(M,N,N+1)$ is open in the relative topology generated by $\mathcal{SC}(M,N)$.*

PROOF. Assume that $M \geq N \geq 2$.

(1) We prove the first assertion by establishing that the complement $\mathcal{F}\backslash\mathcal{SC}(m)$ is open, that is, if $\Phi = \{\varphi_k\}_{k=1}^M \in \mathcal{F}$ is a frame which is not $m-$scalable, we prove that there exists $\varepsilon > 0$ such that for any frame $\Psi = \{\psi_k\}_{k=1}^M \in \mathcal{F}$ for which
$$\|\varphi_k - \psi_k\| < \varepsilon \quad \text{for all } k \in [M],$$
we have that Ψ is not $m-$scalable. Thus assume that $\Phi = \{\varphi_k\}_{k=1}^M$ is a frame which is not $m-$scalable and define the finite set $\mathcal{I}$ of subsets by
$$\mathcal{I} := \{I \subset [M] : \#I = m\}.$$
By Proposition 3.10, for each $I \in \mathcal{I}$ there exists $y_I \in \bigcap_{k\in I} H^+(F(\varphi_k))$, that is, $\min_{k\in I}\langle y_I, F(\varphi_k)\rangle > 0$. By the continuity of the map F, there exists $\varepsilon > 0$ such that for each $\{\psi_k\}_{k=1}^M \subset \mathbb{R}^N$ with $\|\psi_k - \varphi_k\| < \varepsilon$ for all $k \in [M]$ we still have $\min_{k\in I}\langle y_I, F(\psi_k)\rangle > 0$. We can choose $\varepsilon > 0$ sufficiently small to guarantee that $\Psi = \{\psi_k\}_{k=1}^M \in \mathcal{F}$. Again from Proposition 3.10 we conclude that Ψ is not $m-$scalable for any $N \le m \le M$. Hence, $\mathcal{SC}(m)$ is closed in $\mathcal{F}$.

(2) To prove the first part of the second assertion, we note that by Proposition 2.3, $\mathcal{SC}(N+1, N, N) = \complement_{\mathcal{SC}(N+1,N)}^{\mathcal{SC}_+(N+1,N,N+1)}$, which, as proved above is closed in $\mathcal{F}(N+1, N)$. Thus, $\mathcal{SC}_+(N+1, N, N+1)$ is open in the relative topology of $\mathcal{SC}(N+1, N)$. (In fact, since $\mathcal{SC}(N+1, N)$ is closed in $\mathcal{F}(N+1, N)$, then $\mathcal{SC}_+(N+1, N, N+1)$ is also open in $\mathcal{F}(N+1, N)$.)

For the last part, we proceed as follows. Assume $M \ge N + 1$ and let $\Phi \in \mathcal{SC}_+(M, N, N+1)$. Then, there exists $I \subset [M]$, $\#I = N + 1$ such that Φ_I is stricily $(N + 1)-$scalable. Thus, $\Phi_I \in \mathcal{SC}_+(N + 1, N, N + 1)$ which is open in $\mathcal{SC}(N + 1, N)$. This implies that there exists $\varepsilon > 0$ such that for any $\Psi_I = \{\psi_k\}_{k\in I} \in \mathcal{SC}(N + 1, N)$, with $\|\psi_k - \varphi_k\| < \varepsilon$, then $\Psi_I \in \mathcal{SC}_+(N + 1, N, N + 1)$. Thus for any $\Psi \in \mathcal{SC}(M, N)$ with $\|\psi_k - \varphi_k\| < \varepsilon$, for $k \in [M]$, we have $\Psi \in \mathcal{SC}_+(M, N, N + 1)$.

□

The next theorem is the main result of this section. It shows that the set of scalable frames is nowhere dense in the set of frames if M is not "too large" with respect to N.

THEOREM 4.2. *Let $M \ge N \ge 2$. If $M < d + 1 = N(N + 1)/2$ then $\mathcal{SC}(M, N)$ does not contain interior points. In other words, for the boundary of $\mathcal{SC}(M, N)$ we have*
$$\partial\mathcal{SC}(M, N) = \mathcal{SC}(M, N).$$

For the proof of Theorem 4.2 we shall need two lemmas. Recall that for a frame $\Phi = \{\varphi_k\}_{k=1}^M \in \mathcal{F}$ we use the notation
$$X_\Phi = \{\varphi_i\varphi_i^T : i \in [M]\}.$$

LEMMA 4.3. *Let $\{\varphi_k\}_{k=1}^M \subset \mathbb{R}^N$ be such that $\dim\operatorname{span} X_\Phi < \frac{N(N+1)}{2}$. Then there exists $\varphi_0 \in \mathbb{R}^N$ with $\|\varphi_0\| = 1$ such that $\varphi_0\varphi_0^T \notin \operatorname{span} X_\Phi$.*

PROOF. Assume the contrary. Then each rank-one orthogonal projection is an element of $\operatorname{span} X_\Phi$. But by the spectral decomposition theorem every symmetric matrix in $\mathbb{R}^{N\times N}$ is a linear combination of such projections. Hence, $\operatorname{span} X_\Phi$

coincides with the linear space S_N of all symmetric matrices in $\mathbb{R}^{N\times N}$. Therefore,

$$\dim \operatorname{span} X_\Phi = \frac{N(N+1)}{2},$$

which is a contradiction. □

The following lemma shows that for a generic M-element set $\{\varphi_i\}_{i=1}^M \subset \mathbb{R}^N$ (or matrix in $\mathbb{R}^{N\times M}$, if the φ_i are considered as columns) the subspace $\operatorname{span} X_\Phi$ has the largest possible dimension.

LEMMA 4.4. *Let $D := \min\{M, N(N+1)/2\}$. Then the set*

$$\left\{\Phi \in \mathbb{R}^{N\times M} : \dim \operatorname{span} X_\Phi = D\right\}$$

is dense in $\mathbb{R}^{N\times M}$.

PROOF. Let $\Phi = \{\varphi_i\}_{i=1}^M \in \mathbb{R}^{N\times M}$ and $\varepsilon > 0$. We will show that there exists $\Psi = \{\psi_i\}_{i=1}^M \in \mathbb{R}^{N\times M}$ with $\|\Phi - \Psi\| < \varepsilon$ and $\dim \operatorname{span} X_\Psi = D$. For this, set $\mathcal{W} := \operatorname{span} X_\Phi$ and let k be the dimension of $\mathcal{W}$. If $k = D$, nothing is to prove. Hence, let $k < D$. Without loss of generality, assume that $\varphi_1\varphi_1^T, \ldots, \varphi_k\varphi_k^T$ are linearly independent. By Lemma 4.3 there exists $\varphi_0 \in \mathbb{R}^N$ with $\|\varphi_0\| = 1$ such that $\varphi_0\varphi_0^T \notin \mathcal{W}$. For $\delta > 0$ define the symmetric matrix

$$S_\delta := \delta\left(\varphi_{k+1}\varphi_0^T + \varphi_0\varphi_{k+1}^T\right) + \delta^2\varphi_0\varphi_0^T.$$

Then there exists *at most* one $\delta > 0$ such that $S_\delta \in \mathcal{W}$ (regardless of whether $\varphi_{k+1}\varphi_0^T + \varphi_0\varphi_{k+1}^T$ and $\varphi_0\varphi_0^T$ are linearly independent or not). Therefore, we find $\delta > 0$ such that $\delta < \varepsilon/M$ and $S_\delta \notin \mathcal{W}$. Now, for $i \in [M]$ put

$$\psi_i := \begin{cases} \varphi_i & \text{if } i \neq k+1 \\ \varphi_{k+1} + \delta\varphi_0 & \text{if } i = k+1 \end{cases}$$

and $\Psi := \{\psi_i\}_{i=1}^M$. Let $\lambda_1, \ldots, \lambda_{k+1} \in \mathbb{R}$ such that

$$\sum_{i=1}^{k+1} \lambda_i \psi_i \psi_i^T = 0.$$

Then, since $\psi_{k+1}\psi_{k+1}^T = \varphi_{k+1}\varphi_{k+1}^T + S_\delta$, we have that

$$\lambda_{k+1} S_\delta = -\sum_{i=1}^{k+1} \lambda_i \varphi_i \varphi_i^T \in \mathcal{W},$$

which implies $\lambda_{k+1} = 0$ and therefore also $\lambda_1 = \ldots = \lambda_k = 0$. Hence, we have $\dim \operatorname{span} X_\Psi = k+1$ and $\|\Phi - \Psi\| < \varepsilon/M$. If $k = D-1$, we are finished. Otherwise, repeat the above construction at most $D - k - 1$ times. □

REMARK 4.5. For the case $M \geq N(N+1)/2$, Lemma 4.4 has been proved in [**CC**, Theorem 2.1]. In the proof, the authors note that X_Φ spans S_N if and only if the frame operator of X_Φ (considered as a system in S_N) is invertible. But the determinant of this operator is a polynomial in the entries of φ_i, and the complement of the set of roots of such polynomials is known to be dense.

Proof of Theorem 4.2. Assume the contrary. Then, by Lemma 4.4, there even exists an interior point $\Phi = \{\varphi_i\}_{i=1}^M \in \mathcal{SC}(M,N)$ of $\mathcal{SC}(M,N)$ for which the linear space $\mathcal{W} := \operatorname{span} X_\Phi$ has dimension M. Since Φ is scalable, there exist $c_1, \ldots, c_M \geq 0$ such that

$$\sum_{i=1}^{M} c_i \varphi_i \varphi_i^T = \mathrm{Id}\,.$$

Without loss of generality we may assume that $c_1 > 0$.

By Lemma 4.3 there exists $\varphi_0 \in \mathbb{R}^N$ with $\|\varphi_0\| = 1$ such that $\varphi_0 \varphi_0^T \notin \mathcal{W}$. As in the proof of Lemma 4.4, we set

$$S_\delta := \delta\left(\varphi_1 \varphi_0^T + \varphi_0 \varphi_1^T\right) + \delta^2 \varphi_0 \varphi_0^T.$$

Then, for $\delta > 0$ sufficiently small, we have that $S_\delta \notin \mathcal{W}$ and $\Psi := \{\varphi_1 + \delta\varphi_0, \varphi_2, \ldots, \varphi_M\} \in \mathcal{SC}(M,N)$. Hence, there exist $c_1', \ldots, c_M' \geq 0$ such that

$$\sum_{i=1}^{M} c_i \varphi_i \varphi_i^T = \mathrm{Id} = c_1'(\varphi_1 + \delta\varphi_0)(\varphi_1 + \delta\varphi_0)^T + \sum_{i=2}^{M} c_i' \varphi_i \varphi_i^T = \sum_{i=1}^{M} c_i' \varphi_i \varphi_i^T + c_1' S_\delta.$$

This implies $c_1' S_\delta \in \mathcal{W}$, and thus $c_1' = 0$. But then we have

$$c_1 \varphi_1 \varphi_1^T + \sum_{i=2}^{M} (c_i - c_i') \varphi_i \varphi_i^T = 0,$$

which yields $c_1 = 0$ as $\varphi_1 \varphi_1^T, \ldots, \varphi_M \varphi_M^T$ are linearly independent. A contradiction. □

The boundary of $\mathcal{SC}(M,N)$ when $M \geq N(N+1)/2$ seems to be a little complicated to describe. To illustrate this, we consider the case $N = 2$, and we characterize $\partial\mathcal{SC}(M,2)$ when $M = 3$ and when $M \geq 4$. The key to understanding this two dimensional setting is the fact that the map F given in (3.3) has a clear geometrical meaning. We observe that when $N = 2$, F takes the form

$$F\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} x^2 - y^2 \\ xy \end{bmatrix} \quad \text{for } \begin{bmatrix} x \\ y \end{bmatrix} \in \mathbb{R}^2.$$

Proposition 4.6. *$\Phi = \{\varphi_k\}_{k=1}^3 \in \partial\mathcal{SC}(3,2)$ if and only if $\Phi \in \mathcal{SC}(3,2,2)$ and there exists $k_0 \in \{1,2,3\}$ such that $\{\varphi_k\}_{k=1,k\neq k_0}^3$ is an orthogonal set and $\varphi_{k_0} \in \operatorname{co}(\{\varphi_k\}_{k=1,k\neq k_0}^3)$.*

Proof. Recall that from Proposition 4.1 we proved that the set $\mathcal{SC}(3,2,2)$ is closed in $\mathcal{SC}(3,2)$ (and in $\mathcal{F}(3,2)$), and $\mathcal{SC}_+(3,2,3)$ is an open set in $\mathcal{SC}(3,2)$. Moreover, since $\mathcal{SC}(3,2) = \mathcal{SC}(3,2,2) \cup \mathcal{SC}_+(3,2,3)$ with $\mathcal{SC}(3,2,2) \cap \mathcal{SC}_+(3,2,3) = \varnothing$, we conclude that $\partial\mathcal{SC}(3,2) \subset \mathcal{SC}(3,2,2)$. Moreover, if $\Phi = \{\varphi_k\}_{k=1}^3 \in \partial\mathcal{SC}(3,2)$, and for each ℓ, $\varphi_\ell \notin \operatorname{co}(\{\varphi_k\}_{k=1,k\neq\ell}^3)$. Then, Φ contains an orthogonal set and there exists a neighborhood of Φ that contains only scalable frames. Which contradicts the fact that $\Phi \in \partial\mathcal{SC}(3,2)$. Hence $\partial\mathcal{SC}(3,2)$ is contained in the set described in the proposition.

Now if $\Phi = \{\varphi_k\}_{k=1}^3$ satisfies the hypothesis of the proposition, then without loss of generality we can assume that $\{\varphi_k\}_{k=1}^2$ is the standard ONB in $\mathbb{R}^2$. Then every neighborhood of Φ contains frames that are scalable and frames that are not scalable. □

When the number of frame vectors $M \geq 4$ (equivalently, the redundancy $M/2 \geq 2$), the boundary of $\mathcal{SC}(M,2)$ has a different structure.

PROPOSITION 4.7. *Let $M \geq 4$, then*

$$\partial\mathcal{SC}(M,2) = \partial\mathcal{SC}(M,2,3) = \mathcal{SC}(M,2,2) \cap \mathsf{C}_{\mathcal{SC}(M,2)}^{\mathcal{SC}_+(M,2,3)}.$$

PROOF. Recall that for $m \geq 3$, $\mathcal{SC}(M,2,m) = \mathcal{SC}(M,2,3)$, and $\mathcal{SC}(M,2) = \mathcal{SC}(M,2,2) \cup \mathcal{SC}(M,2,3)$. In addition, we can assume without loss of generality that all frames in $\mathcal{SC}(M,2)$ are in the upper-half plane.

Let $\Phi \in \mathcal{SC}(M,2,2) \cap \mathsf{C}_{\mathcal{SC}(M,2)}^{\mathcal{SC}_+(M,2,3)}$. Thus Φ contains an orthogonal set and without any loss of generality we may assume that $\{\varphi_1, \varphi_2\}$ is the standard ONB in $\mathbb{R}^2$. Because, $M \geq 4$, and $\Phi \notin \mathcal{SC}_+(M,2,3)$, we see that no frame vector of Φ will lie in the second quadrant. Consequently, $\varphi_k \in \mathrm{co}(\{\varphi_1, \varphi_2\})$, for $k = 3, \ldots, M$ i.e., all other frame vectors are in the first quadrant. It then follows that every neighborhood of Φ contains frames that are scalable and frames that are not scalable. Hence, $\Phi \in \partial\mathcal{SC}(M,2)$. Consequently, $\mathcal{SC}(M,2,2) \cap \mathsf{C}_{\mathcal{SC}(M,2)}^{\mathcal{SC}_+(M,2,3)} \subset \partial\mathcal{SC}(M,2)$.

By Proposition 4.1 $\mathcal{SC}_+(M,2,3)$ is open in $\mathcal{SC}(M,2)$. Hence, if $\Phi \in \mathcal{SC}_+(M,2,3)$, then $\Phi \notin \partial\mathcal{SC}(M,2)$. Thus, $\partial\mathcal{SC}(M,2) \subset \mathsf{C}_{\mathcal{SC}(M,2)}^{\mathcal{SC}_+(M,2,3)}$.

Similarly, if $\Phi \in \mathsf{C}_{\mathcal{SC}(M,2)}^{\mathcal{SC}(M,2,2)}$, then $\Phi \in \mathcal{SC}_+(M,2,3)$, which implies that $\Phi \notin \partial\mathcal{SC}(M,2)$. Consequently, $\partial\mathcal{SC}(M,2) \subset \mathcal{SC}(M,2,2)$. Consequently, $\partial\mathcal{SC}(M,2) = \mathcal{SC}(M,2,2) \cap \mathsf{C}_{\mathcal{SC}(M,2)}^{\mathcal{SC}_+(M,2,3)}$ □

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