Stochastic Processes in Neuroscience
Part II

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CHAPTER 5

Asymptotic Behavior and Stability

In the last two chapters we stochastic differential equations and first models of oscillators under the influence of noise describing one single neuron. For such equations it is rarely the case that we can obtain explicit solution formulas, thus an exact (pathwise) description of the dynamics is challenging. However, most of the "interesting" local behavior occurs in the neighborhood of **equilibrium points** and it is of interest how small or large perturbations in the initial condition or due to noise effect the behavior of solutions. One question might be if solutions stay near equilibrium points or if they even tend to move closer as time elapses. Morally speaking, this is the subject of **stability theory**.

Let us now be more specific and consider for nonlinear $b : \mathbb{R}^d \to \mathbb{R}^d$, $d \geq 1$, the system of autonomous ordinary differential equations

$$\frac{d}{dt} X_t = b\big(X_t\big). \tag{5.1}$$

A point $x^* \in \mathbb{R}^d$ is called equilibrium point of (5.1), if $b(x^*) = 0$. In particular, $X_t = x^*$ is a constant solution to (5.1) with initial value $X_0 = x^*$. We define several types of stability in the next section and also introduce the relatively elementary notion of linear stability the method of **linearized stability analysis** for equations of type (5.1). Since the scope of this simple method is rather restricted we also study a nonlinear method due to **Lyapunov**, which has its counterpart for stochastic differential equations. In the last part we generalize these concepts in some sense and study the the law of solutions and their asymptotic behavior, which leads to the notion of **invariant measures**. Note that the scope of this chapter is to provide a toolbox for applications rather than to introduce a detailed, general theory on these topics.

5.1. Linearized Stability Analysis

First, we introduce a formal definition of the concept of stability, actually a slightly more general one than we need at this point.

Definition 5.1. An equilibrium point x^* is called

(1) **stable**, if for all $\varepsilon > 0$ there exists $\delta > 0$ such that

$$\|X_0 - x^*\| < \delta \quad \Rightarrow \quad \|X_t - x^*\| < \varepsilon \quad \text{for all } t \geq 0,$$

(2) **locally asymptotically stable**, if it is stable and locally attracting, i. e. there exists $\delta > 0$ such that

$$\|X_0 - x^*\| < \delta \quad \Rightarrow \quad \lim_{t\to\infty} \|X_t - x^*\| = 0,$$

(3) **globally asymptotically stable**, if it is also globally attracting, i. e.

$$\lim_{t\to\infty} \|X_t - x^*\| = 0 \quad \text{for all } X_0 \in \mathbb{R}^d,$$

(4) **locally/globally exponentially stable**, if there exist constants $M > 0$, $\kappa^* > 0$ such that

$$\|X_t - x^*\| \leq M\mathrm{e}^{-\kappa^* t}\|X_0 - x^*\|$$

for all $t \geq 0$ and X_0 from the δ-ball around x^* and $X_0 \in \mathbb{R}^d$ in the global case, respectively,

(5) **unstable**, if it is not stable.

Remark 5.2. The stability from i. is also called Lyapunov stability and it fits for a more general framework introduced in the next section. By definition of the concepts above one can see immediately that

loc./glob. exponentially stable $\Rightarrow$ loc./glob. asymptotically stable $\Rightarrow$ stable.

For an illustration of the method called linear stability analysis we go back to a system of linear ordinary differential equations given by

$$\frac{d}{dt}X_t = AX_t \in \mathbb{R}^d. \tag{5.2}$$

In this case it is obvious, that $x^* = 0$ is an equilibrium point of (5.2) and we want to study its stability depending on the properties of the matrix A.

Theorem 5.3. *Let $x^* = 0$ be the equilibrium point of* (5.2).

(1) *$x^* = 0$ is stable, iff all eigenvalues have a non-positive real part and for all eigenvalues with real part equal to 0 the eigenspace has dimension 1.*
(2) *$x^* = 0$ is unstable, iff there exists an eigenvalue with positive real part or real part equal to 0 and eigenspace of dimension ≥ 2.*
(3) *$x^* = 0$ is locally asymptotically stable, iff all eigenvalues have negative real part.*

Proof. We only prove the theorem in the special case of A having diagonal form. Then, we can easily compute the solution to (5.2), which is $X_t = \mathrm{e}^{tA}X_0$, and in terms of the orthonormal basis $\{e_k\}$ and the eigenvalues $\{\lambda_k\}$ of A the following holds

$$\|X_t - x^*\| = \|\mathrm{e}^{tA}X_0\| \leq \Big(\sum_k |X_0 \cdot e_k|\Big) \max_k \|\mathrm{e}^{tA}e_k\| = \sqrt{d}\|X_0\| \max_k \mathrm{e}^{\lambda_k t}.$$

If there exist k with $\operatorname{Re}\lambda_k > 0$ this obviously tends to infinity as $t \to \infty$. Also in the case of iii. we have that $\|X_t\| \to 0$ as $t \to \infty$. If there exists k with $\operatorname{Re}\lambda_k = 0$, then $\|X_t\| \leq \sqrt{d}\|X_0\|$ and given $\varepsilon > 0$ the choice of $\delta = \varepsilon/\sqrt{d}$ yields stability. $\square$

There is an easy consequence concerning the equivalence of the different notions of stability in Definition 5.1.

Corollary 5.4. *The following assertions are equivalent.*

(1) *All eigenvalues of A have negative real parts.*
(2) *$x^* = 0$ is locally asymptotically stable.*
(3) *$x^* = 0$ is globally exponentially stable with rate κ^* given by the maximum of all of A's eigenvalues' real parts.*

Now let us focus again on the nonlinear ordinary differential equation (5.1) where $b : \mathbb{R}^d \to \mathbb{R}^d$ is globally Lipschitz continuous, i. e. there exists $L > 0$ such that $\|b(x) - b(y)\| \leq L\|x - y\|$ for all $x, y \in \mathbb{R}^d$. It is well-known that for every initial value x there exists a unique (possibly explosive) solution X_t with $X_0 = x$. Let x^* be an equilibrium point for this system and suppose in addition that b is also continuously differentiable in x^*. Then by definition and with $b(x^*) = 0$

$$b(x^* + h) = b(x^*) + Db(x^*)h + R(h) = Db(x^*)h + R(h)$$

where Db denotes the Jacobian and the error terms R are of higher order, i. e. $R(h) \in o(\|h\|)$. In a small neighborhood around x^* it is reasonable to look at the linearized equation for $Y_t = X_t - x^*$

$$\frac{d}{dt}Y_t = JY_t, \quad J = Db(x^*). \tag{5.3}$$

The main objective is to compare the dynamics of (5.1) with the simpler one (5.3), of course always only in a small neighborhood of the equilibrium point x^*. It is often convenient to set $x^* = 0$ in such results, which can always be achieved by a shift in the phase space. With the help of the stability results for linear systems we can state the main result below for equilibrium points with an additional property.

Definition 5.5. An equilibrium point x^* of (5.1) is called **hyperbolic** if the matrix J has no eigenvalues with zero real part.

Theorem 5.6 (Linearized Stability). *Let $x^* = 0$ be a hyperbolic equilibrium point for* (5.1)*. Then, x^* is also an equilibrium point for its linearization* (5.3) *and it is either locally exponentially stable for* (5.1)*, if all eigenvalues of J have negative real parts, or unstable if at least one eigenvalue has positive real part.*

Proof. Exercise. □

Remark 5.7. Note that the Theorem 5.6 remains silent on the issue of what happens if some eigenvalues have zero real parts while all others are negative. In these cases, the analysis of the linearization is not enough and nonlinear effects are responsible for stability or instability of the equilibrium point. In particular for high dimensional systems such a scenario does not happen in rare cases.

5.2. Lyapunov Stability for Ordinary Differential Equations

The scope of the linearized stability analysis we just introduced is the behavior of a system near an equilibrium point when a linear approximation is adequate. It does not however tell us anything when nonlinear effects dominate or about what happens farther away from equilibrium. For this reason we consider a nonlinear method due to Lyapunov which is based on so-called Lyapunov functions that—morally speaking—have x^* as a global minimum and are monotone decreasing along trajectories of X_t.

The following special class of (5.1) is useful to get a good grasp of Lyapunov's idea. Consider b being the gradient of a potential $V \in C^2(\mathbb{R}^d; \mathbb{R})$, i. e.

$$\frac{d}{dt}X_t = -\nabla V\big(X_t\big), \tag{5.4}$$

where $\nabla V = (\partial_{x_1} V, \ldots, \partial_{x_d} V)$ denotes the gradient of V. In this case all equilibrium points x^* are zeros of ∇V, in particular they are local extrema. Now suppose that

V has exactly one global minimum x^* and that $\nabla V(x) \neq 0$ for all $x \neq x^*$. We can now look at the time evolution of $V(X_t)$ and it follows that

$$\frac{d}{dt}V\big(X_t\big) = \langle \nabla V\big(X_t\big), \frac{d}{dt}X_t\rangle = -\|\nabla V\big(X_t\big)\|^2 < 0 \quad \text{if } X_t \neq x^*,$$

where $\langle x, y\rangle$ denotes the usual inner product in $\mathbb{R}^d$. If in addition V is uniformly strictly convex—since $V \in C^2$ this implies the Hessian of V is uniformly positive definite—it follows by Taylor's formula that

$$\langle \nabla V(x), x - x^*\rangle = \langle \nabla V(x^*) + H_V(\xi)(x - x^*), x - x^*\rangle \geq \kappa^* \|x - x^*\|^2$$

for all $x \in \mathbb{R}^d$, where $\kappa^* > 0$, $\xi \in \{x^* + \theta(x - x^*) : \theta \in [0,1]\}$ and $H_V := (\partial_{x_i x_j} V)_{ij}$ denotes the Hessian of V. This implies that the Euclidean norm squared is monotone decreasing in t, even more

$$\frac{d}{dt}\|X_t - x^*\|^2 = 2\langle X_t - x^*, \frac{d}{dt}X_t\rangle = -2\langle \nabla V\big(X_t\big), X_t - x^*\rangle \leq -2\kappa^*\|X_t - x^*\|^2.$$

With the product rule it follows that

$$\|X_t - x^*\|^2 \leq \mathrm{e}^{-2\kappa^* t}\|X_0 - x^*\|^2,$$

i. e. x^* is globally exponentially stable with rate κ^*.

The idea of Lyapunov was to mimic the roles of V and the Euclidean norm for general dynamical systems to obtain a quantity, which is monotone decreasing in time. In the following we assume again w. l. o. g. that $x^* = 0$.

Definition 5.8. Let U be some open neighborhood of $x^* = 0$ and $V \in C^1(U, [0, \infty))$. V is called a **Lyapunov function** for (5.1), if $\langle \nabla V(x), b(x)\rangle \leq 0$ and V is positive definite, i. e. $V(0) = 0$ and $V(x) > 0$ for every non-zero $x \in U$. V is called a **strict Lyapunov function**, if moreover $\langle \nabla V(x), b(x)\rangle < 0$ for every non-zero $x \in U$.

Theorem 5.9 (Lyapunov Stability). *Let V be a Lyapunov function for* (5.1), *then the equilibrium point $x^* = 0$ is stable.*

Proof. Since U is an open neighborhood around 0 we can choose $\varepsilon > 0$ such that the closure of the ε-ball $B_\varepsilon(0)$ around 0 is still contained in U. Set $m := \min_{\|x\|=\varepsilon} V(x)$. Since V is continuous and positive definite, m is positive and we can find $\delta > 0$ such that $0 < \max_{\|x\|<\delta} V(x) < m$. Now let $X_0 \in B_\delta(0)$, then $V(X_0) < m$ and

$$\frac{d}{dt}V\big(X_t\big) = \langle \nabla V\big(X_t\big), b\big(X_t\big)\rangle \leq 0 \quad \Rightarrow \quad V\big(X_t\big) \leq V\big(X_0\big) \tag{5.5}$$

for every $t \geq 0$. Thus, $V(X_t) < m$ and $\|X_t\| \neq \varepsilon$ for all $t \geq 0$. Since we started inside the ε-ball around 0 this shows $\|X_t\| < \varepsilon$ and therefore $x^* = 0$ is stable. □

Theorem 5.10 (Asymptotic Stability). *Let V be a strict Lyapunov function, then $x^* = 0$ is locally asymptotically stable. Moreover, if $U = \mathbb{R}^d$ and V is coercive, i. e. $\lim_{\|x\|\to\infty} V(x) = +\infty$, then $x^* = 0$ is globally asymptotically stable.*

Proof. Consider the first statement, then by Theorem 5.9, $x^* = 0$ is stable. W. l. o. g. we can assume V can be extended to the closure $\overline{U}$ such that all its properties still hold. Stability guarantees that there exists $\delta > 0$ such that if $X_0 \in B_\delta(0)$ we have $X_t \in \overline{U}$ for all times. It remains to prove that $X_t \to 0$ as $t \to \infty$.

Now suppose $V(X_t)$ does not converge to 0 as $t \to \infty$. By (5.5) $V(X_\cdot)$ is decreasing and since V is positive definite and continuous there has to be $c > 0$ such that $V(X_t) \geq c$ for every $t \geq 0$ and also

$$\inf_{t \geq 0} \|X_t\| \geq r$$

for some $r > 0$. By compactness of $\overline{U} \setminus B_r(0)$ the continuous function $\langle \nabla V(X_t), b(X_t) \rangle$ attains its extrema, hence it is bounded away from 0. This implies $\frac{d}{dt} V(X_t) \leq -k$ for some $k > 0$, which contradicts the non-negativity of V.

So far, we have proven via contradiction that $V(X_t) \to 0$ as $t \to \infty$. Given $\varepsilon > 0$ we can define

$$v_\varepsilon := \min_{x \in \overline{U} \setminus B_\varepsilon(0)} V(x) > 0$$

by compactness, continuity and positive definiteness. Obviously, there exists $T > 0$ such that $V(X_t) < v_\varepsilon$ for all $t \geq T$. This implies $X_t \in B_\varepsilon(0)$ since it cannot leave U due to stability. As ε was arbitrary, asymptotic stability follows.

The second statement remains an exercise. □

In order to obtain exponential stability we have to restrict ourselves to a subclass of Lyapunov functions as defined below.

Definition 5.11. A function $V \in C^1(U, [0, \infty))$ is called a **quadratic Lyapunov function** for (5.1), if there exist constants $\alpha_1, \alpha_2, \alpha_3 > 0$ such that

$$\alpha_1 \|x\|^2 \leq V(x) \leq \alpha_2 \|x\|^2 \quad \text{and} \quad \langle \nabla V(x), b(x) \rangle \leq -\alpha_3 \|x\|^2$$

for all $x \in U$.

Theorem 5.12 (Exponential Stability). *The equilibrium point $x^* = 0$ is locally exponentially stable with $\kappa^* = \frac{\alpha_3}{2\alpha 2}$ and $M = \sqrt{\frac{\alpha_2}{\alpha_1}}$, if there exists a quadratic Lyapunov function for* (5.1).

Proof. Exercise. □

As a concluding remark concerning the two discussed methods one might say that since it is not always straightforward to obtain a suitable Lyapunov function, the method of linearized stability analysis seems easier to apply. However as a fact, whenever linearization works we can always find a $V(x) = \|x\|^2$, where $\|\cdot\|$ is some equivalent norm such that $\langle x, Jx \rangle \leq -c\|x\|^2$ holds for some $c > 0$ and all $x \in \mathbb{R}^d$. On the contrary, there are very simple ($d = 1$) examples with non-hyperbolic equilibrium points for which the linearization approach is useless.

Example 5.13. Consider $d = 2$ and $b(x, y) = (-x^3 + 2y^3, -2xy^2)$. The point $(0, 0)$ is an equilibrium point but apparently a non-hyperbolic one, since all eigenvalues are 0. Thus, we have to find a suitable Lyapunov function and reasonable (and often useful) guess is the square of the Euclidean norm or more general $V(x, y) := \alpha x^2 + \beta xy + y^2$ with α, β to be determined. We immediately see that this choice of V is positive definite on $U = \mathbb{R}^2$. In our case, we stick with $V(x) = x^2 + y^2$, hence

$$\begin{aligned}
\langle \nabla V(x, y), b(x, y) \rangle &= 2x\big(-x^3 + 2y^3\big) + 2y\big(-2xy^2\big) \\
&= -2x^4 + 4xy^3 - 4xy^3 = -2x^4 \leq 0.
\end{aligned}$$

Theorem 5.9 yields stability for $(0,0)$. For asymptotic stability or even exponential stability one might try to look for a different choice of V (see exercises), but another method called **LaSalle's Invariance Principle** allows to prove asymptotic stability in such cases.

DEFINITION 5.14. (1) A subset $S \subset \mathbb{R}^d$ is called the **ω-limit set** of the solution X to (5.1) with initial value X_0, if for every $y \in S$ we can find a sequence $t_n \nearrow \infty$ such that $X_{t_n} \to y$ as $n \to \infty$. One also denotes $S = \omega(X_0)$.
(2) A subset $S \subset \mathbb{R}^d$ is called a **(forward) invariant set**, if for all $y \in S$ and $X_0 = y$ we have $X_t \in M$ for all $t \geq 0$.

REMARK 5.15. In other words, the ω-limit set is the set of all accumulation points of a given trajectory with start in X_0. Clearly, any ω-limit set is forward invariant.

THEOREM 5.16 (LaSalle's Invariance Principle). *Suppose there exists a neighborhood U of $x^* = 0$ and a positive definite function $V : U \to \mathbb{R}$ with $\langle \nabla V(x), b(x) \rangle \leq 0$ for all $x \in U$. Define*

$$S := \{x \in U : \langle \nabla V(x), b(x) \rangle = 0\}.$$

Then, there exists $\delta > 0$ such that for all $X_0 \in B_\delta(0)$ the trajectory X_t converges to the largest invariant set contained in S as $t \nearrow \infty$, in particular $\omega(X_0)$ is contained in the largest invariant set in S.

COROLLARY 5.17. *If $x^* = 0$ is the only invariant set in S, then it is locally asymptotically stable.*

COROLLARY 5.18. *If $U = \mathbb{R}^d$, V is coercive and $x^* = 0$ is the only invariant set in S, then it is globally asymptotically stable.*

EXAMPLE 5.19. In Example 5.13 the Lyapunov function V was only positive semi-definite. Thus, we cannot apply Theorem 5.10 to obtain asymptotic stability. However, the set S of points where $\langle \nabla V(x), b(x) \rangle = 0$ is small, in our case the y-axis, and the vector field b is not parallel to this set. Hence, $x^* = 0$ is the only invariant set in S and Theorem 5.16 yields global asymptotic stability.

5.3. Stability for Stochastic Differential Equations

Throughout this section we shall study stability problems for stochastic differential equations of the form

$$\mathrm{d}X_t = b\big(X_t\big)\,dt + \sigma\big(X_t\big)\,dW_t, \quad X_0 = \xi_0, \tag{5.6}$$

with $b : \mathbb{R}^d \to \mathbb{R}^d$, $\sigma : \mathbb{R}^d \to \mathbb{R}^{d\times n}$ globally Lipschitz, W a n-dimensional Brownian motion on $(\Omega, \mathcal{F}, \mathbb{P})$ and ξ_0 $\mathcal{F}$-measurable. For a fixed, deterministic initial condition $x \in \mathbb{R}^d$ we denote by X_t^x the solution to (5.6) at time $t \geq 0$ with $X_0^x = x$.

The question of stability first raises another question, namely if there exists an equilibrium point. One can easily see that in the case of additive noise, i.e. $\sigma = \text{const}$ there is no point x^* where the right hand side vanishes. Thus, we restrict ourselves to multiplicative noise and assume $x^* = 0$ and $b(0) = 0$, $\sigma(0) = 0$ such that the constant process $X_t^0 = 0$ is a solution to (5.6). Recall that the restriction to 0 is not a loss in generality. There are now several possibilites to define stability for stochastic differential equations.

DEFINITION 5.20. The equilibrium $x^* = 0$ is said to be

(1) **stable in probability** if for every $\varepsilon \in (0,1)$, $\delta > 0$ there exists a $r > 0$ such that
$$\mathbb{P}\left[\sup_{t\geq 0} \|X_t^x\| < \delta\right] \geq 1-\varepsilon \quad \text{for all } \|x\| < r,$$
(2) **asymptotically stable in probability** if it is stable in probability and for every $\varepsilon \in (0,1)$ there exists $r > 0$ such that
$$\mathbb{P}\left[\lim_{t\to\infty} \|X_t^x\| = 0\right] \geq 1-\varepsilon \quad \text{for all } \|x\| < r.$$

DEFINITION 5.21. The equilibrium $x^* = 0$ is said to be

(1) p**-stable**, $p > 0$ if for every $\varepsilon > 0$ there exists $r > 0$ such that
$$\sup_{t\geq 0} \mathbb{E}\left[\|X_t^x\|^p\right] < \varepsilon \quad \text{for all } \|x\| < r,$$
(2) **asymptotically** p**-stable** if it is p-stable and moreover
$$\lim_{t\to\infty} \mathbb{E}\left[\|X_t^x\|^p\right] = 0,$$
(3) **exponentially** p**-stable** if there are constants $M, \kappa^* > 0$ such that for all $t \geq 0$
$$\mathbb{E}\left[\|X_t^x\|^p\right] \leq M\|x\|^p \mathrm{e}^{-\kappa^* t}.$$

The special cases $p = 1$ and $p = 2$ are most commonly considered and are referred to as **stability in the mean** and **stability in mean square**, respectively. In the following, we generalize the Lyaponov function approach to stochastic differential equations. Apparently, we first have to find a suitable definition of a Lyaponov function for (5.6) and Itô's formula for $f \in C^2(\mathbb{R}^d;\mathbb{R})$ implies the following.

$$\begin{aligned}
\mathrm{d}f(X_t) &= \Big(\sum_{i=1}^d b_i(X_t)\partial_{x_i} f(X_t) + \frac{1}{2}\sum_{i,j=1}^d a_{ij}(X_t)\partial^2_{x_i x_j} f(X_t)\Big)\, dt \\
&\quad + \sum_{i=1}^d \sum_{j=1}^n \partial_{x_i} f(X_t)\sigma_{ij}(X_t) dW_t^j \\
&= Lf(X_t)dt + dM_t^f
\end{aligned}$$

with

(1) $a = \sigma \cdot \sigma^T$,
(2) the infinitesimal generator $Lf(x) := \frac{1}{2}\sum_{i,j=1}^d a_{ij}(x)\partial^2_{x_i x_j} f(x) + \sum_{i=1}^d b_i(x)\partial_{x_i} f(x)$ associated with (5.6), also called **Kolmogorov operator** and
(3) the local martingale $M_t^f = \int_0^t \sum_{i,j=1}^d \partial_{x_i} f(X_s)\sigma_{ij}(X_s) dW_s^j$.

This point of view is strongly related to the **martingale problem** for stochastic differential equations, see [**SV79**], namely given the infinitesimal generator L associated with (5.6), the process
$$f(X_t) - f(X_0) - \int_0^t Lf(X_s)\, ds = M_t^f$$
is a (local) martingale for every $f \in C^2(\mathbb{R}^d)$.

A reasonable choise for a Lyaponov function V would be a positive definite $V \in C^2(\mathbb{R}^d;\mathbb{R})$ and $LV \leq 0$. However, unlike in the deterministic case, where V is supposed to be sufficiently smooth around $x=0$, in the stochastic case there might not even exist such functions which are smooth in the origin. For this reason let us define the class of functions following [**Kha12**].

DEFINITION 5.22. Let $V : \mathbb{R}^d \to [0,\infty)$ be twice continuously differentiable everywhere, except possibly for $x=0$ and continuous on all closed sets $\{x : \|x\| \geq \varepsilon\}$, $\varepsilon > 0$. V is called a Lyaponov function for (5.6) if $LV \leq 0$ and V is positive definite. Again, V is a strict Lyapunov function if $LV < 0$ for all $x \neq 0$.

LEMMA 5.23. *Let $\beta \in \mathbb{R}$, $x \neq 0$, then*

$$\mathbb{E}\left[\|X_t^x\|^\beta\right] \leq \|x\|^\beta \mathrm{e}^{kt},$$

where k is a constant depending only on β and the global Lipschitz constant of (5.6).

PROOF. let $\varepsilon > 0$, then $f(x) = \|x\|^\beta$ is twice continuously differentiable on $\{x : \|x\| > \varepsilon\}$. Define the stopping time

$$\tau^\varepsilon := \inf\{t \geq 0 : \|X_t^x\| \leq \varepsilon\}$$

and Itô's formula yields

$$\begin{aligned}
\|X_{t\wedge\tau^\varepsilon}^x\|^\beta =& \|x\|^\beta + \beta\int_0^{t\wedge\tau^\varepsilon} \|X_s^x\|^{\beta-2}\Big[\langle b\big(X_s^x\big), X_s^x\rangle + \frac{1}{2}\sum_{i=1}^d a_{ii}\big(X_s^x\big)\Big]\, ds \\
&+ \beta\int_0^{t\wedge\tau^\varepsilon} \|X_s^x\|^{\beta-2}\sum_{j=1}^n \langle\sigma_{\cdot j}\big(X_s^x\big), X_s^x\rangle\, dW_t^j \\
&+ \frac{\beta}{2}(\beta-2)\int_0^{t\wedge\tau^\varepsilon} \|X_s^x\|^{\beta-4}\langle a\big(X_x^s\big)X_s^x, X_s^x\rangle\, ds.
\end{aligned}$$

Now taking expectations and using the global Lipschitz condition together with $b(0)=0$ and $\sigma(0)=0$, we obtain

$$\mathbb{E}\left[\|X_{t\wedge\tau^\varepsilon}^x\|^\beta\right] \leq \|x\|^\beta + k\mathbb{E}\left[\int_0^{t\wedge\tau^\varepsilon} \|X_s^x\|^\beta\, ds\right] = \|x\|^\beta + k\int_0^t \mathbb{E}\left[\|X_{s\wedge\tau^\varepsilon}^x\|^\beta\right]\, ds.$$

Gronwall's lemma implies the estimate

$$\mathbb{E}\left[\|X_{t\wedge\tau^\varepsilon}^x\|^\beta\right] \leq \|x\|^\beta \mathrm{e}^{kt}. \tag{5.7}$$

Also, it is clear that $\{t\wedge\tau^\varepsilon < t\} \subseteq \{\|X_{t\wedge\tau^\varepsilon}^x\| \leq \varepsilon\}$ and therefore with Markov's inequality and (5.7) with $\beta=-1$

$$\mathbb{P}\left[t\wedge\tau^\varepsilon < t\right] \leq \mathbb{P}\left[\|X_{t\wedge\tau^\varepsilon}^x\|^{-1} \geq \varepsilon^{-1}\right] \leq \frac{\varepsilon}{\|x\|}\mathrm{e}^{kt}.$$

Hence

$$\mathbb{P}\left[t\wedge\tau^\varepsilon \to t \text{ as } \varepsilon \to 0\right] = 1 \tag{5.8}$$

and we can combine this with (5.7) to conclude the result. □

With the lemma above we can state the main theorem on stability in probability.

THEOREM 5.24 (Stability). *Let V be a Lyapunov function for* (5.6). *Then $V(X_t)$ converges $\mathbb{P}$-a. s. as $t \to \infty$ and $x^* = 0$ is stable in probability. Moreover for any $\delta > 0$*

$$\lim_{\|x\|\to 0} \mathbb{P}\left[\sup_{t\geq 0} \|X_t^x\| \geq \delta\right] = 0,$$

i.e., we have some form of local stability of $x^* = 0$ with probability 1.

PROOF. Let $\varepsilon > 0$ and τ^ε as in Lemma 5.23. Then we can apply Itô's formula to $V(X^x_{t\wedge\tau^\varepsilon})$

$$V\big(X^x_{t\wedge\tau^\varepsilon}\big) = V\big(X^x_{s\wedge\tau^\varepsilon}\big) + \int_{s\wedge\tau^\varepsilon}^{t\wedge\tau^\varepsilon} LV\big(X^x_r\big)\, dr + M^V_{t\wedge\tau^\varepsilon} - M^V_{s\wedge\tau^\varepsilon},$$

so that by $LV \leq 0$ and the optional sampling theorem

$$\mathbb{E}\left[V\big(X^x_{t\wedge\tau^\varepsilon}\big) \mid \mathcal{F}_s\right] \leq V\big(X^x_{s\wedge\tau^\varepsilon}\big),$$

where $\{\mathcal{F}_t\}$ is the filtration generated by the Brownian motion W. In particular, $V(X^x_{t\wedge\tau^\varepsilon})$ is a non-negative supermartingale. Letting $\varepsilon \to 0$ it follows by (5.8) that $\mathbb{P}\left[\tau^0 < \infty\right] = 0$, thus also $V(X^x_t)$ is a non-negative supermartingale and the almost sure martingale convergence theorem implies that the limit $\lim_{t\to\infty} V(X^x_t)$ exists $\mathbb{P}$-a. s.

For the proof of stability, let $\delta > 0$ be given and define the stopping time

$$\tau_\delta := \inf\{t \geq 0 : \|X^x_t\| > \delta\}.$$

Obviously,

$$\mathbb{P}\left[\sup_{0\leq s\leq t} \|X^x_s\| > \delta\right] = \mathbb{P}\left[\tau_\delta \leq t\right]$$

and furthermore

$$\mathbb{E}\left[V\big(X^x_{t\wedge\tau_\delta}\big)\right] \geq \mathbb{E}\left[\mathbb{1}_{\{\tau_\delta\leq t\}} V\big(X^x_{\tau_\delta}\big)\right] \geq \mathbb{P}\left[\tau_\delta \leq t\right] \inf_{\|x\|\geq\delta} V(x) =: \mathbb{P}\left[\tau_\delta \leq t\right] V_\delta.$$

Thus, using Markov's inequality and the supermartingale property we have shown that

$$\mathbb{P}\left[\sup_{0\leq s\leq t} \|X^x_s\| > \delta\right] \leq V_\delta^{-1}\mathbb{E}\left[V\big(X^x_{t\wedge\tau_\delta}\big)\right] \leq V_\delta^{-1} V(x)$$

and since $V(0) = 0$ and V continuous this implies the result by letting $t \to \infty$. □

THEOREM 5.25 (Asymptotic Stability). *Let V be a strict Lyapunov function for* (5.6). *Then, $x^* = 0$ is asymptotically stable in probability.*

PROOF. Let $\varepsilon, \delta > 0$ be given, then by Theorem 5.24 we can find $r > 0$ such that

$$\mathbb{P}\left[\|X^x_t\| < \delta\right] \geq 1 - \frac{\varepsilon}{4} \tag{5.9}$$

for all $\|x\| < r$. Now choose $0 < \alpha < \beta < \|x\|$ to be precisely determined later. Again, we need the stopping times from above, namely τ^α, the first entry time into the ball of radius α, and τ_δ, the first exit time from the ball of radius δ. Then, Itô's formula implies

$$\mathbb{E}\left[V\big(X^x_{t\wedge\tau^\alpha\wedge\tau_\delta}\big)\right] \leq V(x) + \mathbb{E}\left[t\wedge\tau^\alpha\wedge\tau_\delta\right] \inf_{\|y\|=\alpha} LV(y) =: -\mathbb{E}\left[t\wedge\tau^\alpha\wedge\tau_\delta\right] L_\alpha.$$

By assumption, $L_\alpha > 0$ is well-defined. Markov's inequality implies

$$t\mathbb{P}\left[\tau^\alpha\wedge\tau_\delta \geq t\right] \leq \mathbb{E}\left[t\wedge\tau^\alpha\wedge\tau_\delta\right] \leq L_\alpha^{-1} V(x),$$

hence $\mathbb{P}[\tau^\alpha \wedge \tau_\delta < \infty] = 1$. But we also have $\mathbb{P}[\tau_\delta < \infty] \le \frac{\varepsilon}{4}$ by (5.9) and so $\mathbb{P}[\tau^\alpha < \infty] \ge 1 - \frac{\varepsilon}{4}$, in particular we can find $\theta > 0$ large enough such that $\mathbb{P}[\tau^\alpha < \theta] \ge 1 - \frac{\varepsilon}{2}$.

The rest of the story of the proof is more or less this one: once the process entered the (possibly very small) ball of radius α it does not leave the ball of radius β with probability greater than $1-\varepsilon$. For this purpose define the additional stopping times

$$\sigma := \begin{cases} \tau^\alpha & \text{if } \tau^\alpha < \tau_\delta \wedge \theta \\ \infty & \text{otherwise,} \end{cases}$$

and

$$\tau_\beta := \inf\{t > \sigma : \|X_t^x\| \ge \beta\}.$$

With the supermartingale property follows $\mathbb{E}\left[V(X_{t\wedge\tau_\beta}^x)\right] \le \mathbb{E}\left[V(X_{t\wedge\sigma}^x)\right]$ and in particular we can restrict this to the subset where σ is finite, i. e.

$$\mathbb{E}\left[\mathbb{1}_{\{\tau^\alpha<\tau_\delta\wedge\theta\}} V(X_{t\wedge\tau_\beta}^x)\right] \le \mathbb{E}\left[\mathbb{1}_{\{\tau^\alpha<\tau_\delta\wedge\theta\}} V(X_{t\wedge\sigma}^x)\right].$$

We also know that $\{\tau_\beta < \infty\}$ is a subset of this event and therefore

$$V_\beta \mathbb{P}[\tau_\beta < \infty] := \inf_{\|y\|=\beta} V(y)\mathbb{P}[\tau_\beta < \infty] \le \sup_{\|y\|=\alpha} V(y) =: V^\alpha.$$

At this point we determine the choice of α, namely small enough such that $V_\beta^{-1}V^\alpha \le \frac{\varepsilon}{4}$ and then of course follows $\mathbb{P}[\tau_\beta < \infty] \le \frac{\varepsilon}{4}$. We can put the pieces together as follows.

$$\begin{aligned} \mathbb{P}[\sigma < \infty \text{ and } \tau_\beta = \infty] &\ge \mathbb{P}[\tau^\alpha < \tau_\delta \wedge \theta] - \mathbb{P}[\tau_\beta < \infty] \\ &\ge \mathbb{P}[\tau^\alpha < \theta] - \mathbb{P}[\tau_\delta < \infty] - \mathbb{P}[\tau_\beta < \infty] \\ &\ge 1 - \frac{\varepsilon}{2} - \frac{\varepsilon}{4} - \frac{\varepsilon}{4} = 1 - \varepsilon. \end{aligned}$$

Since β was arbitrary this concludes the proof. □

THEOREM 5.26 (Exponential p-Stability). *The equilibrium $x^* = 0$ is exponentially p-stable if there exists a Lyapunov function V satisfying*

$$\alpha_1\|x\|^p \le V(x) \le \alpha_2\|x\|^p \quad \textit{and} \quad LV(x) \le -\alpha_3\|x\|^p$$

for all $x \in \mathbb{R}^d$ and constants $\alpha_1, \alpha_2, \alpha_3 > 0$.

PROOF. Exercise. □

EXAMPLE 5.27. As an example we consider a stochastic differential equation with explicitly known solution since then we can compare the abstract stability results with the properties of the solution formula. Our choice is the geometric Brownian motion in, i. e.

$$\mathrm{d}X_t = bX_t\,dt + \sigma X_t\,dW_t$$

in dimension $d = 1$. A smooth Lyapunov function as in the previous section for deterministic systems would be $V(x) = x^2$. Obviously $LV(x) = (\sigma^2 + 2b)x^2 \le 0$ iff $b < -\frac{\sigma^2}{2}$. Theorem 5.24 yields stability in probability but in comparison to the ordinary differential equation with $\sigma = 0$ the noise seems to destabilize. However, this is not the case and in comparison with the explicit solution we see why.

$$\begin{aligned} X_t^x &= \exp\Big(bt + \sigma W_t - \frac{\sigma^2}{2}t\Big)x \\ &= \exp\Big(\Big(b - \frac{\sigma^2}{2}\Big)t + \sigma W_t\Big)x \to 0 \end{aligned}$$

$\mathbb{P}$-a. s. even for $b < \frac{\sigma^2}{2}$, in particular $b > 0$ in which the solution of the corresponding ordinary differential equation explodes. We can see this stability with a different choice for the Lyapunov function, namely $V(x) = |x|^{1-\frac{2b}{\sigma^2}} =: |x|^{2\alpha}$ for $b < \frac{\sigma^2}{2}$ which is not differentiable in 0 for $b > -\frac{\sigma^2}{2}$. With this choice of V it follows

$$LV(x) = \alpha|x|^{2\alpha}\Big(2b + \sigma^2(2\alpha - 1)\Big) = 0$$

since

$$2b + \sigma^2(2\alpha - 1) = 2b + \sigma^2\left(1 - \frac{2b}{\sigma^2} - 1\right) = 0\,,$$

and we obtain even asymptotic stability in probability by Theorem 5.25. Also, Theorem 5.26 yields exponential p-stability for $p < 1 - \frac{2b}{\sigma^2}$.

5.4. Invariant Measures for Stochastic Differential Equations

In this section we depart from the pathwise picture of the solution to a stochastic differential equation and study how its law evolves as time passes by. A natural analogue to an equilibrium point, i. e. pathwise stationarity, is a stationary law. Since the law of a solution to a stochastic differential equation evolves according to its transition probabilities, stationarity is described as invariance with respect to the corresponding transition semigroup. We introduce these notions in detail below and then study existence of such laws using the Krylov-Bogoliubov theory.

So again we consider throughout the whole section a stochastic differential equation of the form (5.6), for convenience stated again below,

$$\mathrm{d}X_t = b\big(X_t\big)\,dt + \sigma\big(X_t\big)\,dW_t, \quad X_0 = \xi_0. \tag{5.10}$$

We also assume a global Lipschitz condition on b and σ and recall that X_t^x denotes the solution to (5.10) with deterministic initial condition $X_0 = x \in \mathbb{R}^d$. With this in mind we can define the transition probabilities

$$p_t(x, A) := \mathbb{P}\left[X_t^x \in A\right], \quad t \geq 0, \quad A \in \mathcal{B}(\mathbb{R}^d).$$

Furthermore note that $W_u^s := W_u - W_s$, $u \geq s$, is again a Brownian motion with start in 0 and its natural filtration is given by

$$\mathcal{F}_s^t := \sigma\Big(\big\{W_u - W_s : u \in [s,t]\big\}\Big), \quad 0 \leq s \leq t.$$

With our assumptions the existence of a unique strong solution $X_t^{s,x}$, $t \geq s$ of

$$\mathrm{d}X_t = b\big(X_t\big)\,dt + \sigma\big(X_t\big)\,dW_t^s, \quad X_s = x \tag{5.11}$$

is guaranteed. Moreover, uniqueness implies the so-called **flow-property**, i. e. for $0 \leq s \leq t \leq u$ we have that

$$X_u^{s,x} = X_u^{t,X_t^{s,x}}.$$

Proposition 5.28 (Markov Property). *Let $\mathcal{F}_t = \sigma(\{W_s,\ s \in [0,t]\})$ be the natural filtration of W. Then the solution of* (5.10) *satisfies*

$$\mathbb{P}[X_t \in A \mid \mathcal{F}_s] = \mathbb{P}[X_t \in A \mid X_s] = p_{t-s}(X_s, A) \quad \mathbb{P}\text{-}a.\,s.$$

Proof. Clearly we can write (5.10) in its integral form in the following way

$$\begin{aligned} X_t &= x + \int_0^t b\big(X_u\big)\,du + \int_0^t \sigma\big(X_u\big)\,\mathrm{d}W_u \\ &= X_s + \int_s^t b\big(\tilde{X}_u\big)\,du + \int_s^t \sigma\big(\tilde{X}_u\big)\,\mathrm{d}W_u^s \end{aligned}$$

where $\tilde{X}_u$, $u \geq s$, is the unique (strong) solution of (5.11) with $\tilde{X}_s = X_s$, so that by the flow property

$$\tilde{X}_u = X_u^{s,X_s} \quad \text{and thus} \quad X_t = X_t^{s,X_s}.$$

Now, $X_t^{s,x}$ only depends on the Brownian increment $W_u^s = W_u - W_s$, $u \geq s$, which is independent of $\mathcal{F}_s$, so that $X_t^{s,x}$ is independent of $\mathcal{F}_s$, too. Consequently, for F $\mathcal{F}_s$-measurable,

$$\begin{aligned} \mathbb{E}\left[F\,\mathbb{P}[X_t \in A \mid \mathcal{F}_s]\right] &= \mathbb{E}\left[F\,\mathbb{E}\left[\mathbb{1}_A\big(X_t\big) \mid \mathcal{F}_s\right]\right] = \mathbb{E}\left[F\,\mathbb{E}\left[\mathbb{1}_A\Big(X_t^{s,X_s}\Big) \mid \mathcal{F}_s\right]\right] \\ &= \mathbb{E}\left[F\,\mathbb{E}\left[\mathbb{1}_A\Big(X_t^{s,X_s}\Big)\right]\right] = \mathbb{E}\left[F\,\mathbb{P}[X_t \in A \mid X_s]\right]. \end{aligned}$$

Furthermore the map $(s,x) \mapsto X_t^{s,x}$ is Borel-measurable (proof via Picard-Lindelöf iteration) and thus $(s,x) \mapsto \mathbb{P}[X_t^{s,x} \in A]$, $A \in \mathcal{B}(\mathbb{R}^d)$ is Borel-measurable. It follows that $p_{t-s}(X_s, A) = \mathbb{P}[X_t \in A \mid X_s]$ is $\sigma(X_s)$-measurable, hence $p_{t-s}(X_s, A)$ is a version of $\mathbb{P}[X_t \in A \mid X_s]$. □

The proof of Proposition 5.28 shows that $p_t(x, dy)$ is a transition kernel of probability measures (stochastic kernel) from $\mathbb{R}_+ \times \mathbb{R}^d$ to $\mathcal{B}(\mathbb{R}^d)$, i.e.,

(1) $A \mapsto p_t(x, A)$ is a probability measure on $(\mathbb{R}^d, \mathcal{B}(\mathbb{R}^d))$ for all $(t,x) \in \mathbb{R}_+ \times \mathbb{R}^d$ and
(2) $(t,x) \mapsto p_t(x, A)$ is $\mathcal{B}(\mathbb{R}_+) \otimes \mathcal{B}(\mathbb{R}^d)$-measurable for all $A \in \mathcal{B}(\mathbb{R}^d)$.

Thus, we can look at the following integral operator on the space of bounded Borel-measurable functions given by

$$\big(P_t f\big)(x) := \int_{\mathbb{R}^d} f(y) p_t(x, dy) = \mathbb{E}\left[f\big(X_t^x\big)\right], \quad x \in \mathbb{R}^d.$$

P_t is well-defined for f Borel-measurable and bounded or non-negative. Furthermore it satisfies the **Chapman-Kolmogorov equation**, namely $P_t \circ P_s = P_{t+s}$ for all $t, s \geq 0$ and $P_0 = \mathrm{Id}$. In particular, $\{P_t\}$ is a semigroup of linear operators. This can be seen quite easily using the Markov property

$$\begin{aligned} \Big(\big(P_t \circ P_s\big) f\Big)(x) &= \Big(P_t\big(P_s f\big)\Big)(x) = \mathbb{E}\left[P_s f\big(X_t^x\big)\right] \\ &= \mathbb{E}\left[\mathbb{E}\left[f\big(X_{t+s}^x\big) \mid \mathcal{F}_t\right]\right] = \mathbb{E}\left[f\big(X_{t+s}^x\big)\right] = \big(P_{t+s} f\big)(x). \end{aligned}$$

Example 5.29. Let us illustrate these objects and properties with a few examples.

(1) Consider Brownian Motion with start in x, i.e. $\mathrm{d}X_t = dW_t$, $X_0 = x$, then we know that

$$p_t(x, A) = \mathbb{P}[x + W_t \in A] = \mathcal{N}(x, t)(A).$$

For the semigroup property observe that $X_{t+s} = x + W_{t+s} = x + (W_{t+s} - W_s) + W_s$ with two independent increments. Hence

$$\begin{aligned}
\big(P_{t+s}f\big)(x) &= \mathbb{E}\left[f\big(x + W_{t+s}\big)\right] = \mathbb{E}\left[\mathbb{E}\left[f\big(x + W_{t+s} - W_s + W_s\big) \mid W_s\right]\right] \\
&= \mathbb{E}\left[\int_{\mathbb{R}} f\big(x + y + W_s\big)\mathcal{N}(0,t)(\,dy)\right] \\
&= \int_{\mathbb{R}}\int_{\mathbb{R}} f(x + y + z)\mathcal{N}(0,s)(\,dz)\mathcal{N}(0,t)(\,dy) \\
&= \int_{\mathbb{R}}\left(\int_{\mathbb{R}} f(z)\mathcal{N}(x+y,s)(\,dz)\right)\mathcal{N}(0,t)(\,dy) \\
&= \Big(P_t\big(P_s f\big)\Big)(x) = \Big(\big(P_t \circ P_s\big)f\Big)(x).
\end{aligned}$$

(2) Consider the Ornstein-Uhlenbeck process $\mathrm{d}X_t = -bX_t\,dt + dW_t$, $X_0 = x$ with $b > 0$. The solution is given by the stochastic convolution

$$X_t = \mathrm{e}^{-bt}x + \int_0^t \mathrm{e}^{-b(t-s)}\,dW_s,$$

where the law of the stochastic integral is given by $\mathcal{N}(0, q_t)$, $q_t = \int_0^t \mathrm{e}^{-2bs}\,ds$. Thus, the transition kernel is

$$p_t(x, A) = \mathcal{N}(\mathrm{e}^{-bt}x, q_t)(A).$$

One can calculate the semigroup property in a similar fashion as above.

$$\begin{aligned}
\Big(\big(P_t \circ P_s\big)f\Big)(x) &= \mathbb{E}\left[P_s f\big(X_t^x\big)\right] \\
&= \int_{\mathbb{R}} P_s f\big(\mathrm{e}^{-bt}x + y\big)\mathcal{N}\big(0, q_t\big)(\,dy) \\
&= \int_{\mathbb{R}}\int_{\mathbb{R}} f\Big(\mathrm{e}^{-bs}\big(\mathrm{e}^{-bt}x + z\big) + y\Big)\mathcal{N}\big(0, q_s\big)(\,dz)\mathcal{N}\big(0, q_t\big)(\,dy) \\
&= \int_{\mathbb{R}}\int_{\mathbb{R}} f\Big(\mathrm{e}^{-b(t+s)}x + \big(\mathrm{e}^{-bs}z + y\big)\Big)\mathcal{N}\big(0, q_s\big)(\,dz)\mathcal{N}\big(0, q_t\big)(\,dy) \\
&= \int_{\mathbb{R}} f\big(\mathrm{e}^{-b(t+s)}x + \tilde{y}\big)\mathcal{N}\big(0, \mathrm{e}^{-2bs}q_t + q_s\big)(\,\mathrm{d}\tilde{y}) = \big(P_{t+s}f\big)(x).
\end{aligned}$$

The last line is due to the fact that the convolution of two normal distributions is again a normal distribution with variance equal to the sum of the variances. Moreover $q_{t+s} = \mathrm{e}^{-2bs}q_t + q_s$.

In the following we fix a semigroup $\{P_t\}$ of stochastic integral operators on $\mathbb{R}^d$.

Definition 5.30. A probability measure μ on $\mathcal{B}(\mathbb{R}^d)$ is called **invariant** for $\{P_t\}$ if

$$\int P_t f\,d\mu = \int f\,d\mu \quad \text{for all } f \in \mathcal{B}_b\big(\mathbb{R}^d\big),$$

i. e. all bounded Borel-measurable functions f. Intuitively this means that if $\xi_0 \sim \mu$ the unique (strong) solution X_t of (5.10) has distribution μ for all times and thus the distribution of X_t is invariant with respect to time.

In the following we study existence of invariant measures for stochastic differential equations of type (5.10) using the **Krylov-Bogoliubov** theory. One key ingredient to this theory is Prokhorov's theorem, which relates the notions of tightness and relative weak compactness for probability measures.

Definition 5.31. (1) A family of probability measures $\Gamma \subseteq \mathcal{M}_1(\mathbb{R}^d)$ is called **tight** if for all $\varepsilon \in (0,1)$ there exists a compact $K_\varepsilon \subseteq \mathbb{R}^d$ such that $\mu\big(K_\varepsilon^C\big) \leq \varepsilon$ for all $\mu \in \Gamma$.
(2) A family of probability measures $\Gamma \subseteq \mathcal{M}_1(\mathbb{R}^d)$ is called **relatively weakly compact** if any sequence (μ_n) in Γ has a weakly convergent subsequence.

Theorem 5.32 (Prokhorov). *The following statements are equivalent:*
(1) $\Gamma \subseteq \mathcal{M}(\mathbb{R}^d)$ *is weakly relatively compact.*
(2) $\Gamma \subseteq \mathcal{M}(\mathbb{R}^d)$ *is tight.*

The main idea to prove existence of an invariant measure is now to find a sequence of measures, which is tight and therefore by Theorem 5.32 has a weakly convergent subsequence. The limit of this subsequence is then a candidate for the invariant measure. The sequence of measures we consider are called **mean occupation times** and defined as follows. Let $T > 0$, $x \in \mathbb{R}$, then

$$\mu_{T,x}(A) := \mathbb{E}\left[\frac{1}{T}\int_0^T \mathbb{1}_A\big(X_t^x\big)\,dt\right], \quad A \in \mathcal{B}\big(\mathbb{R}^d\big)$$

is a probability measure, where the integral yields the total amount of time spent in the set A up to time T, or in other words the **occupation time of** A.

Remark 5.33.

$$\int f\,d\mu_{T,x} = \mathbb{E}\left[\frac{1}{T}\int_0^T f\big(X_t^x\big)\,dt\right] = \frac{1}{T}\int_0^T \mathbb{E}\left[f\big(X_t^x\big)\right] = \frac{1}{T}\int_0^T \big(P_t f\big)(x)\,dt$$

Theorem 5.34 (Krylov-Bogoliubov). *Suppose there exists $x \in \mathbb{R}^d$ for which $\{\mu_{T,x}\}_{T\geq 1}$ is tight. Then there exists an invariant measure μ for $\{P_t\}$.*

Proof. As mentioned above, Prokhorov's theorem implies the existence of a weakly converging subsequence to a limit measure $\mu \in \mathcal{M}_1(\mathbb{R}^d)$, i.e. there exists $T_n \nearrow \infty$, such that $\lim_{n\to\infty} \mu_{T_n,x} = \mu$ weakly. This is equivalent to

$$\lim_{n\to\infty}\frac{1}{T_n}\int_0^{T_n}\big(P_t f\big)(x)\,dt = \int f\,d\mu$$

for all $f \in C_b(\mathbb{R}^d)$. Let $s > 0$ be arbitrary. Then $P_s f \in C_b(\mathbb{R}^d)$, hence

$$\begin{aligned}
\int P_s f\,d\mu &= \lim_{n\to\infty}\frac{1}{T_n}\int_0^{T_n}\Big(P_t\big(P_s f\big)\Big)(x)\,dt = \lim_{n\to\infty}\frac{1}{T_n}\int_0^{T_n}\big(P_{t+s}f\big)(x)\,dt \\
&= \lim_{n\to\infty}\frac{1}{T_n}\int_s^{T_n+s}\big(P_t f\big)(x)\,dt \\
&= \lim_{n\to\infty}\left[\frac{1}{T_n}\int_0^{T_n}\big(P_t f\big)(x)\,dt + \frac{1}{T_n}\left(\int_{T_n}^{T_n+s}\big(P_t f\big)(x)\,dt - \int_0^s \big(P_t f\big)(x)\,dt\right)\right].
\end{aligned}$$

The latter term converges to 0 as $T_n \nearrow \infty$ and the first one converges to $\int f\,d\mu$ as noticed above. Thus, μ is invariant for $\{P_t\}$. □

EXAMPLE 5.35. In this example we illustrate a simple criterion that indeed implies tightness and therefore existence of an invariant measure. Suppose that for some $p > 0$ and $x \in \mathbb{R}^d$, the pth moment of $\|X_t^x\|$ is uniformly bounded, i. e.

$$\mathbb{E}\left[\|X_t^x\|^p\right] \leq C$$

for all $t \geq 0$. Then the family $\{\mu_{T,x}\}$ is tight and thus there exists an invariant measure μ for $\{P_t\}$. This is proven as follows. Let $R > 0$, then

$$\begin{aligned}\mu_{T,x}\left(\overline{B_R}^C\right) &= \int_{\{\|x\|>R\}} d\mu_{T,x} \leq \frac{1}{R^p}\int \|x\|^p \, d\mu_{T,x} \\ &= \frac{1}{R^p}\frac{1}{T}\int_0^T \mathbb{E}\left[\|X_t^x\|^p\right] dt \leq \frac{C}{R^p}\end{aligned}$$

Now for arbitrary $\varepsilon \in (0,1)$ we can choose $R_\varepsilon = (C/\varepsilon)^{\frac{1}{p}}$ such that $\mu_{T,x}(B_{R_\varepsilon}^C) \leq \varepsilon$ for all $T \geq 1$. Since the closed ball of radius R_ε is compact in $\mathbb{R}^d$, $\{\mu_{T,x}\}$ is tight.

As an illustration consider the Ornstein-Uhlenbeck process

$$\mathrm{d}X_t = BX_t \, dt + C \, dW_t$$

with $B, C \in \mathbb{R}^{d\times d}$ and W a d-dimensional Brownian motion. By the variation of constants method the solution is given by

$$X_t^x = \mathrm{e}^{tB}x + \int_0^t \mathrm{e}^{(t-s)B}C \, dW_s,$$

which is $\mathcal{N}(\mathrm{e}^{tB}x, Q_t)$ distributed. Here,

$$Q_t := \int_0^t \mathrm{e}^{sB}C \cdot C^T \mathrm{e}^{sB^T} \, ds$$

By Itô's formula we obtain

$$\mathbb{E}\left[\|X_t^x\|^2\right] = \text{ trace } Q_t.$$

The trace is uniformly bounded in t if the operator norm $\|\mathrm{e}^{sB}\|_{L(\mathbb{R}^d)} \leq e^{-s\kappa_*}$ for some $\kappa_* > 0$, in particular this is the case if B only has strictly negative eigenvalues.

In the following, we use this criterion to deduce existence of an invariant measure for nonlinear equations (5.10) with a recurrent drift term, i.e. with a drift term $b(x)$ pointing back towards the origin for large $\|x\|$. This is a common assumption in models for neurons.

PROPOSITION 5.36. *Suppose that the coefficents of* (5.10) *satisfy*

(1) $\langle b(x), x\rangle \leq \alpha - \lambda\|x\|^2$ *and*
(2) *trace* $(\sigma(x)\cdot\sigma(x)^T) \leq 2C$

for some $\alpha, \lambda, C > 0$*. Then for any* $x \in \mathbb{R}^d$

$$\mathbb{E}\left[\|X_t^x\|^2\right] \leq \mathrm{e}^{-2\lambda t}\|x\|^2 + \frac{\alpha + C}{\lambda}\left(1 - \mathrm{e}^{-2\lambda t}\right),$$

in particular the second moment is uniformly bounded in t*. Hence, there exists an invariant measure for* $\{P_t\}$*.*

PROOF. Itô's formula implies

$$\begin{aligned}\mathrm{d}\|X_t^x\|^2 &= 2\langle X_t^x, \mathrm{d}X_t^x\rangle + \text{ trace } \left(\sigma\left(X_t^x\right)\cdot\sigma\left(X_t^x\right)^T\right) dt \\ &\leq 2\left(\alpha - \lambda\|X_t^x\|^2\right) dt + 2C\, dt + 2\langle X_t^x, \sigma\left(X_t^x\right) dW_t\rangle.\end{aligned}$$

Therefore, using the product rule

$$\mathrm{d}\left(\mathrm{e}^{2\lambda t}\|X_t^x\|^2\right) \leq 2(\alpha + C)\mathrm{e}^{2\lambda t} dt + 2\mathrm{e}^{2\lambda t}\langle X_t^x, \sigma\left(X_t^x\right) dW_t\rangle,$$

hence

$$\mathrm{e}^{2\lambda t}\|X_t^x\|^2 \leq \|x\|^2 + 2(\alpha + C)\int_0^t \mathrm{e}^{2\lambda s}\, ds + 2\int_0^t \mathrm{e}^{2\lambda s}\langle X_s^x, \sigma\left(X_s^x\right) dW_s\rangle$$

Taking expectations on both sides yields the result. □

EXAMPLE 5.37 (FHN-system with Noise). Consider the FitzHugh-Nagumo system with the usual parameters perturbed by independent Brownian motion in both variables.

$$\mathrm{d}\begin{bmatrix} V_t \\ W_t \end{bmatrix} = \begin{bmatrix} f\left(V_t\right) - W_t + I \\ \varepsilon\left(V_t - \gamma W_t\right)\end{bmatrix} dt + \begin{bmatrix} \sigma_{V,V} & \sigma_{V,W} \\ \sigma_{W,V} & \sigma_{W,W}\end{bmatrix}\left(V_t, W_t\right) \mathrm{d}\begin{bmatrix} B_t^V \\ B_t^W \end{bmatrix}. \tag{5.12}$$

We assume that the trace of the diffusion coefficient is bounded as

$$\text{trace } \left(\sigma\cdot\sigma^T\right)(v,w) = \left(\sigma_{VV}^2 + \sigma_{VW}^2 + \sigma_{WV}^2 + \sigma_{WW}^2\right)(v,w) \leq 2C$$

uniformly in v, w. Denote the drift by $b(v,w)$, then

$$\begin{aligned}\langle b(v,w), \begin{bmatrix} v \\ w\end{bmatrix}\rangle &= f(v)v - vw + Iv + \varepsilon vw - \varepsilon\gamma w^2 \\ &\leq -\frac{v^4}{2} + \left(\frac{2+a^2}{2} + \frac{(\varepsilon-1)^2}{2\varepsilon\gamma}\right)v^2 - \frac{\varepsilon\gamma}{2}w^2 + \frac{1}{2}\, I^2 \leq \alpha - \lambda(v^2 + w^2)\end{aligned}$$

for some $\lambda > 0$ and $\alpha > 0$ sufficiently large. To obtain the second line we used the estimates

$$\begin{aligned} f(v)v = -v^4 + (1+a)v^3 - av^2 &\leq -\frac{v^4}{2} + \frac{1+a^2}{2}v^2, \\ (\varepsilon - 1)vw &\leq \frac{\varepsilon\gamma}{2}w^2 + \frac{(\varepsilon-1)^2}{2\varepsilon\gamma}v^2, \\ Iv &\leq \frac{1}{2}I^2 + \frac{1}{2}v^2.\end{aligned}$$

Proposition 5.36 now implies that

$$\mathbb{E}\left[\left\|\begin{bmatrix} V_t \\ W_t\end{bmatrix}\right\|^2\right] \leq \mathrm{e}^{-2\lambda t}\left\|\begin{bmatrix} v_0 \\ w_0\end{bmatrix}\right\|^2 + \frac{\alpha + C}{\lambda}\left(1 - \mathrm{e}^{-2\lambda t}\right)$$

is uniformly bounded in t. Hence there exists an invariant measure μ for (5.12).

It is difficult to get further information on μ via the Krylov-Bogoliubov theory. However, in special cases the structure of the drift allows to explicitly specify the invariant measure. The following example illustrates this for the **Andronov-Hopf oscillator**.

EXAMPLE 5.38. Consider the following stochastic differential equation

$$\mathrm{d}Z_t = b\big(Z_t\big)\,dt + \sigma\big(Z_t\big)\,dW_t, \tag{5.13}$$

with $Z_t = (X_t, Y_t) \in \mathbb{R}^2$, W a two-dimensional Brownian motion and drift given by

$$b\big(X_t, Y_t\big) = \begin{bmatrix} X_t - Y_t - X_t\|Z_t\|^2 \\ Y_t + X_t - Y_t\|Z_t\|^2 \end{bmatrix}$$

Note that $\langle b(z), z\rangle = \|z\|^2 - \|z\|^4 \le 1 - \frac{1}{2}\|z\|^2$, hence for bounded trace $(\sigma\cdot\sigma^T)$ there exists an invariant measure. In the particular case $\sigma = \sigma_0\mathrm{Id}$, $\sigma_0 > 0$ constant, the (unique) invariant measure is given by

$$\mu(\,dz) = Z^{-1}\exp\Big(\frac{1}{\sigma_0^2}\|z\|^2 - \frac{1}{2\sigma_0^2}\|z\|^4\Big)\,dz$$

where

$$Z = \int_{\mathbb{R}^2}\exp\Big(\frac{1}{\sigma_0^2}\|z\|^2 - \frac{1}{2\sigma_0^2}\|z\|^4\Big)\,dz$$

is the normalizing constant. It suffices to check that for any $f \in C_0^2(\mathbb{R}^2)$ we have $\int Lf\,d\mu = 0$, a weaker notion often called **infinitesimal invariance**. Here, L is the Kolmogorov operator associated with (5.13)

$$\big(Lf\big)(z) = \frac{\sigma_0^2}{2}\big(\partial_{xx}f + \partial_{yy}f\big)(z) + \langle b(z), \nabla f(z)\rangle.$$

We can verify this infinitesimal invariance with a simple integration by parts. For this reason, let us first look at the partial derivatives of the density of μ but because it is an exponential we only need the derivate of the exponent. Clearly

$$\partial_x\Big(\frac{1}{\sigma_0^2}\|z\|^2 - \frac{1}{2\sigma_0^2}\|z\|^4\Big) = \frac{2x}{\sigma_0^2} - \frac{2x}{\sigma_0^2}\|z\|^2, \tag{5.14}$$

$$\partial_y\Big(\frac{1}{\sigma_0^2}\|z\|^2 - \frac{1}{2\sigma_0^2}\|z\|^4\Big) = \frac{2y}{\sigma_0^2} - \frac{2y}{\sigma_0^2}\|z\|^2, \tag{5.15}$$

So it follows that

$$\begin{aligned}\int\big(Lf\big)(z)\mu(\,dz) &= \frac{\sigma_0^2}{2}\int\big(\partial_{xx}f + \partial_{yy}f\big)(z) + \langle b(z), \nabla f(z)\rangle\mu(\,dz)\\ &= -\int\langle\begin{bmatrix} x - x\|z\|^2 \\ y - y\|z\|^2\end{bmatrix}, \nabla f(z)\rangle + \langle b(z), \nabla f(z)\rangle\mu(\,dz)\\ &= \int -y\partial_x f(z) + x\partial_y f(z)\mu(\,dz)\end{aligned}$$

using integration by parts and (5.14) and (5.15). Another integration by parts yields

$$= \int\frac{2}{\sigma_0^2}\Big(y\big(x - x\|z\|^2\big) - x\big(y - y\|z\|^2\big)\Big)f(z)\mu(\,dz) = 0.$$

Thus, μ is indeed invariant for (5.13). Its density depends on σ_0 in the following way.

$$Z = \sqrt{2}\pi^{\frac{3}{2}}\sigma_0 \mathrm{e}^{\frac{1}{2\sigma_0^2}}\,\Phi\big(\sigma_0^{-1}\big),$$

where Φ denotes the cumulative density function of the standard normal distribution. Clearly, $Z \to \infty$ as $\sigma_0 \to 0$ so that μ concentrates on a set of Lebesgue measure 0. In fact, it converges to the uniform distribution on the unit circle S^1, which is the invariant measure of the deterministic motion. Figure 1 illustrates this for different values of σ_0 and since μ is rotation invariant a plot of the density at $y = 0$ suffices.

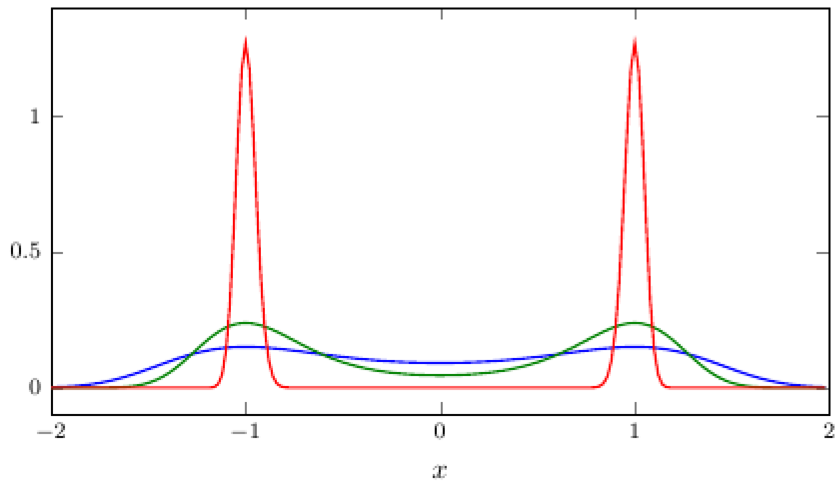

FIGURE 1. A plot of the density of μ at $y = 0$ for three different values of $\sigma_0 = 0.1, 0.55, 1$ with colors red, green and blue, respectively.

CHAPTER 6

Coupled Neurons

Up to now we have only considered the dynamics of a single neuron and have gained a good knowledge of their behavior. In the brain however, the connections between different neurons are as important (or even more) as the single neuron dynamics. As a toy model we consider two neurons that are somehow coupled and thus interacting. In general, neurons are connected by synapses with different properties and their own dynamics described by an additional ordinary differential equation for each synaptic connection. It is reasonable to simplify this to a coupling using linearly scaled voltage differences of the connected neurons. We will do this in this chapter in order to be able to prove results on the behavior of such systems. In particular, a major interest in biophysical applications is the question of how and when the network synchonizes, i. e. their membrane potential coincides and therefore they are spiking synchronously.

6.1. Coupling of Two FHN-Neurons

Recall our primary example, the FitzHugh-Nagumo neuron, and consider a system of two neurons with the same parameters.

$$\begin{aligned} dV_t^i &= \Big(f\big(V_t^i\big) - W_t^i + I\Big)\,dt, \\ dW_t^i &= \epsilon\big(V_t^i - \gamma W_t^i\big)\,dt, \end{aligned}$$

for $i = 1, 2$, where $f(v) = v(1-v)(v-a)$, $a \in (0, \frac{1}{2})$ and $\epsilon, \gamma > 0$. As mentioned in the introduction, it is reasonable to add the coupling via linearly scaled voltage differences, i. e.

$$\theta\big(V_t^j - V_t^i\big), \quad j \neq i$$

as additional input I for neuron i. The resulting system has the following structure:

$$\begin{aligned} dV_t^1 &= \Big(f\big(V_t^1\big) - W_t^1 + I + \theta\big(V_t^2 - V_t^1\big)\Big)\,dt, \\ dW_t^1 &= \epsilon\big(V_t^1 - \gamma W_t^1\big)\,dt, \\ dV_t^2 &= \Big(f\big(V_t^2\big) - W_t^2 + I + \theta\big(V_t^1 - V_t^2\big)\Big)\,dt, \\ dW_t^2 &= \epsilon\big(V_t^2 - \gamma W_t^2\big)\,dt. \end{aligned} \tag{6.1}$$

The parameter θ is called the **coupling strength**. The two neurons are **synchronized** if $V_t^1 = V_t^2$ and $W_t^1 = W_t^2$ and they **synchronize** if $\lim_{t\to\infty} |V_t^1 - V_t^2| + |W_t^1 - W_t^2| = 0$.

In the following, we use the Lyapunov techniques from Chapter 5 to prove that the system (6.1) synchronizes. This can be done by considering the differences $X = (V^1 - V^2, W^1 - W^2)$ with nonlinear drift b defined as the differences of the right hand sides of (6.1) and the equilibrium point $x^* = (0,0)$. Then, synchronization is essentially the asymptotic stability of x^*. Let the parameters be chosen as follows:

$$\theta > \theta^* := \frac{L^*(a)}{2}, \quad L^*(a) = \sup_{\xi \in \mathbb{R}} f'(\xi) = \frac{1 - a + a^2}{3}.$$

The Lyapunov function V of our choice is $V(v^1, w^1, v^2, w^2) := \epsilon(v^1 - v^2)^2 + (w^1 - w^2)^2$, which measures the distance of $(v^1, w^1), (v^2, w^2)$ to a synchronized state. V is indeed a Lyapunov function for (6.1) since

$$\begin{aligned}
&\langle \nabla V\big(v^1, w^1, v^2, w^2\big), b\big(v^1, w^1, v^2, w^2\big)\rangle \\
&= 2\epsilon\Big(f\big(v^1\big) - w^1 + I + \theta\big(v^2 - v^1\big) - f\big(v^2\big) + w^2 - I - \theta\big(v^1 - v^2\big)\Big)\big(v^1 - v^2\big) \\
&\quad + 2\epsilon\big(v^1 - \gamma w^1 - v^2 + \gamma w^2\big)\big(w^1 - w^2\big) \\
&= 2\epsilon\Big(f\big(v^1\big) - f\big(v^2\big)\Big)\big(v^1 - v^2\big) - 4\epsilon\theta\big(v^1 - v^2\big)^2 - 2\epsilon\gamma\big(w^1 - w^2\big)^2 \\
&\le 2\epsilon\big(\gamma^*(a) - 2\theta\big)\big(v^1 - v^2\big)^2 - 2\epsilon\gamma\big(w^1 - w^2\big)^2.
\end{aligned}$$

In the last line we used that

$$\big(f(x) - f(y)\big)(x - y) = \int_0^1 f'(y + s(x - y))\, ds \cdot (x - y)^2 \le \sup_{\xi \in \mathbb{R}} f'(\xi) \cdot (x - y)^2.$$

We also see that for $\theta \ge \theta^*$ V is a Lyapunov function, whereas it is a strict one for all $\theta > \theta^*$. Also, in this case V is a quadratic Lyapunov function in the sense of Definition 5.11 with $\alpha_1 = \epsilon$, $\alpha_2 = 1$ and

$$\alpha_3 = \kappa^* := \min\Big\{2\big(2\theta - L^*(a)\big), 2\epsilon\gamma\Big\}.$$

applied to the seminorm $\|(v^1, w^1, v^2, w^2)\|^2 = V(v^1, w^1, v^{\cdot} w^2)$. Theorem 5.12 then yields the following result.

Proposition 6.1. *Let $\theta > \theta^*$, hence $\kappa^* > 0$. Then*

$$V\big(V_t^1, W_t^1, V_t^2, W_t^2\big) \le \mathrm{e}^{-\kappa^* t} V\big(V_0^1, W_0^1, V_0^2, W_0^2\big).$$

In particular, the dynamical system is synchronizing in the sense that

$$\lim_{t \to \infty} |V_t^1 - V_t^2| + |W_t^1 - W_t^2| = 0$$

and the ***synchronized state*** *$V_t^1 = V_t^2$, $W_t^1 = W_t^2$ is* ***exponentially attracting*** *with* ***exponential rate*** *κ^*.*

Remark 6.2. (1) The proof of the proposition does not require the particular shape of f. In fact the statement remains true if f is any function satisfying the one-sided Lipschitz condition

$$\big(f(x) - f(y)\big)(x - y) \le L^*(x - y)^2.$$

(2) Numerical approximations of (6.1) show that synchronization also occurs for much smaller coupling strength. The mathematical justification will require a much more refined analysis. For very small coupling strength so called **phase locking** is observed, i. e. the two neurons do no longer synchronize but show the same periodic behaviour with a constant but non-zero phase difference.

EXAMPLE 6.3. $a = 0.25$, $\epsilon = 0.01$, $\gamma = 0.95 \cdot \gamma^*(a)$ for $\gamma^*(a) = \frac{3}{1-a+a^2} \approx 3.69$ and $I = 0.06$, thus periodic spiking for the uncoupled system. Numerically, we observe synchronization even for coupling strength below the threshold. For very small θ there appears a phenomenon called **phase locking**.

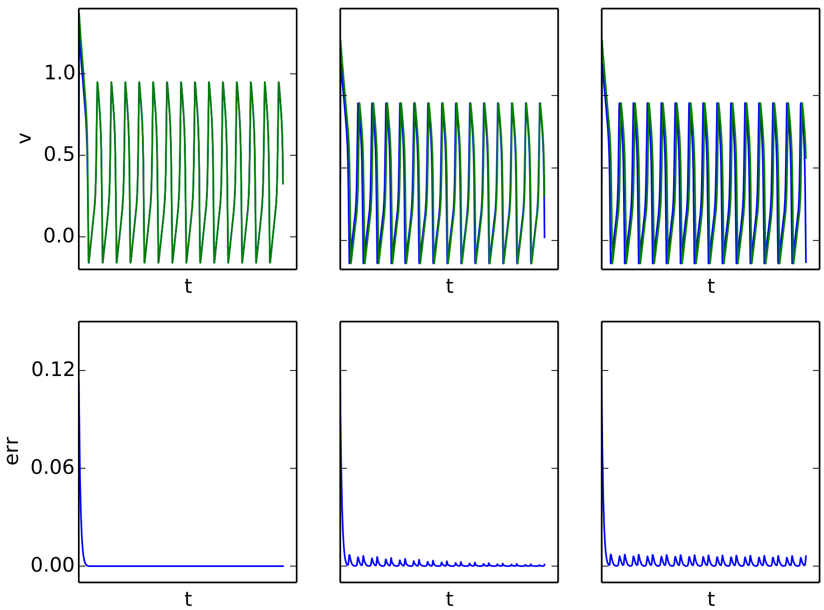

FIGURE 1. A plot of V_t^1 and V_t^2 in blue and green, as well as $\mathrm{err} = \epsilon(V_t^1 - V_t^2)^2 + (W_t^1 - W_t^2)^2$ for three different values $\theta = \theta^*$, $\theta = 0.02 \cdot \theta^*$ and $\theta = 0.001 \cdot \theta^*$ from left to right.

6.2. Two FHN-Neurons with Common Noise

In this part, we analyze the effects of noise on the synchronization of two FitzHugh-Nagumo neurons. We choose the same coupling as well as the same parameters as in the previous section. Additionally, we perturb both neurons by a common additive noise in both variables. The system then looks as follows.

$$
\begin{aligned}
dV_t^1 &= \Big(f\big(V_t^1\big) - W_t^1 + I + \theta\big(V_t^2 - V_t^1\big)\Big)\,dt + \sigma\,dB_t^V \\
dW_t^1 &= \epsilon\big(V_t^1 - \gamma W_t^1\big)\,dt + \sqrt{\epsilon}\sigma\,dB_t^W, \\
dV_t^2 &= \Big(f\big(V_t^2\big) - W_t^2 + I + \theta\big(V_t^1 - V_t^2\big)\Big)\,dt + \sigma\,dB_t^V, \\
dW_t^2 &= \epsilon\big(V_t^2 - \gamma W_t^2\big)\,dt + \sqrt{\epsilon}\sigma\,dB_t^W,
\end{aligned}
\tag{6.2}
$$

where B^V, B^W are two independent Brownian motions but drive both both neurons.

PROPOSITION 6.4. *Let $\theta > \theta^*$, hence $\kappa^* > 0$. Then for all ω*

$$V\big(V_t^1, W_t^1, V_t^2, W_t^2\big)(\omega) \leq \mathrm{e}^{-\kappa^* t} V\big(V_0^1, W_0^1, V_0^2, W_0^2\big)(\omega).$$

*In particular, the random dynamical system (6.2) is synchronizing pathwise and the synchronized state $V_t^1 = V_t^2$, $W_t^1 = W_t^2$ is **pathwise exponentially attracting** with **exponential rate** κ^*.*

PROOF. We will see that due to the common noise the difference of the two solution satisfies an ordinary differential equation with random coefficients since the Brownian paths are identical, hence vanish. In detail,

$$\begin{aligned} dV_t^1 - dV_t^2 &= \Big(f\big(V_t^1\big) - f\big(V_t^2\big) - \big(W_t^1 - W_t^2\big) + 2\theta(V_t^2 - V_t^1)\Big)\,dt, \\ dW_t^1 - dW_t^2 &= \epsilon\Big(\big(V_t^1 - V_t^2\big) - \gamma\big(W_t^1 - W_t^2\big)\Big)\,dt. \end{aligned}$$

Thus, we can easily apply Itô's formula to the time-dependent function $\mathrm{e}^{\kappa^* t} V(V_t^1, W_t^1, V_t^2, W_t^2)$ to obtain

$$\begin{aligned} & d\Big(\mathrm{e}^{\kappa^* t} V\big(V_t^1, W_t^1, V_t^2, W_t^2\big)\Big) \\ &= \kappa^* \mathrm{e}^{\kappa^* t} V\big(V_t^1, W_t^1, V_t^2, W_t^2\big) + 2\epsilon\big(V_t^1 - V_t^2\big)\mathrm{e}^{\kappa^* t}\big(dV_t^1 - dV_t^2\big) \\ &\quad + 2\big(W_t^1 - W_t^2\big)\mathrm{e}^{\kappa^* t}\big(dW_t^1 - dW_t^2\big) \\ &\quad + \epsilon\big(d\langle V^1\rangle_t + d\langle V^2\rangle_t - 2d\langle V^1, V^2\rangle_t\big) \\ &\quad + \big(d\langle W^1\rangle_t + d\langle W^2\rangle_t - 2d\langle W^1, W^2\rangle_t\big) \\ &\leq \epsilon\big(\kappa^* - 2\theta + \gamma^*(a)\big)\big(V_t^1 - V_t^2\big)^2 \mathrm{e}^{\kappa^* t}\,dt \\ &\quad + \big(\kappa^* - 2\epsilon\gamma\big)\big(W_t^1 - W_t^2\big)^2 \mathrm{e}^{\kappa^* t}\,dt \\ &\quad + \epsilon \mathrm{e}^{\kappa^* t}\big(\sigma^2\,dt + \sigma^2\,dt - 2\sigma^2\,dt\big) \\ &\quad + \mathrm{e}^{\kappa^* t}\big(\epsilon\sigma^2\,dt + \epsilon\sigma^2\,dt - 2\epsilon\sigma^2\,dt\big) \leq 0. \end{aligned}$$

Consequently,

$$\mathrm{e}^{\kappa^* t} V\big(V_t^1, W_t^1, V_t^2, W_t^2\big)(\omega) \leq V\big(V_0^1, W_0^1, V_0^2, W_0^2\big)(\omega)$$

for all ω, which implies the assertion. □

Compared to Proposition 6.1, Proposition 6.4 does not seem to differ that much. However, as seen in the numerical simulations, the theoretical results are not optimal and there is sychronization for smaller coupling strenghts. On the contrary, in the case of a common additive noise we observe synchronization for "arbitrary small" coupling, also in cases where the deterministic counterpart reveals phase locking.

EXAMPLE 6.5. $a = 0.25$, $\epsilon = 0.01$, $\gamma = 0.95 \cdot \gamma^*(a)$ and $I = 0.06$ as before. For $\theta = 0.001 \cdot \theta^*$, i. e. there is phase locking for the deterministic counterpart, we observe synchronization depending on the noise intensity σ.

6.3. Coupled FHN-Neurons with Independent Noise

In the last section we used a common noise for both neurons, in other words the maximal dependent noise. Now let us look at the other extreme and replace

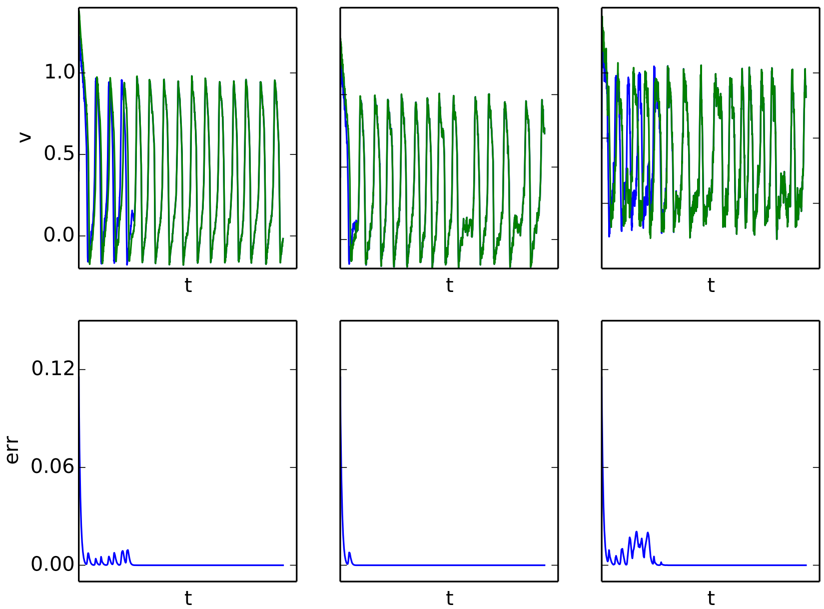

FIGURE 2. A plot of V_t^1 and V_t^2 in blue and green, as well as $\mathrm{err} = \epsilon(V_t^1 - V_t^2)^2 + (W_t^1 - W_t^2)^2$ for three different values $\sigma = 0.01$, $\sigma = 0.02$ and $\sigma = 0.05$ from left to right.

B^V, B^W by independent Brownian motions B^{V^i}, B^{W^i}.

$$
\begin{aligned}
dV_t^1 &= \Big(f\big(V_t^1\big) - W_t^1 + I + \theta\big(V_t^2 - V_t^1\big)\Big)\,dt + \sigma\,dB_t^{V^1} \\
dW_t^1 &= \epsilon\big(V_t^1 - \gamma W_t^1\big)\,dt + \sqrt{\epsilon}\sigma\,dB_t^{W^1}, \\
dV_t^2 &= \Big(f\big(V_t^2\big) - W_t^2 + I + \theta\big(V_t^1 - V_t^2\big)\Big)\,dt + \sigma\,dB_t^{V^2}, \\
dW_t^2 &= \epsilon\big(V_t^2 - \gamma W_t^2\big)\,dt + \sqrt{\epsilon}\sigma\,dB_t^{W^2}.
\end{aligned}
\tag{6.3}
$$

With a similar calculation as above we can prove the following statement.

PROPOSITION 6.6. *Let $\theta > \theta^*$, hence $\kappa^* > 0$. Then*

$$
\mathbb{E}\left[\big(V_t^1, W_t^1, V_t^2, W_t^2\big)\right] \le \mathrm{e}^{-\kappa^* t} V\big(V_0^1, W_0^1, V_0^2, W_0^2\big) + \frac{4\epsilon\sigma^2}{\kappa^*}.
$$

In particular, the differences $V_t^1 - V_t^2$ and $W_t^1 - W_t^2$ of the solution of (6.3) are uniformly bounded in the mean square, i. e.

$$
\sup_{t\ge 0} \mathbb{E}\left[|V_t^1 - V_t^2|^2 + |W_t^1 - W_t^2|^2\right] < \infty.
$$

REMARK 6.7. It is intuitively clear that the system (6.3) cannot synchronize in the sense described above due to independent additive noise. The voltage of both neurons tend towards each other but leave the synchronous state immediately. This is the same issue as described earlier, because the stochastic differential equation for the difference has additive noise, thus no equilibrium point.

PROOF. Itô's formula applied to the same function as above yields

$$\begin{aligned}
&d\Big(\mathrm{e}^{\kappa^* t} V\big(V_t^1, W_t^1, V_t^2, W_t^2\big)\Big) \\
&\le \epsilon\big(\kappa^* - 2\theta + L^*(a)\big)\big(V_t^1 - V_t^2\big)^2 \mathrm{e}^{\kappa^* t}\, dt + \big(\kappa^* - 2\epsilon\gamma\big)\big(W_t^1 - W_t^2\big)^2 \mathrm{e}^{\kappa^* t}\, dt \\
&\quad + \epsilon\big(d\langle V^1\rangle_t + d\langle V^2\rangle_t - 2d\langle V^1, V^2\rangle_t\big) + \big(d\langle W^1\rangle_t + d\langle W^2\rangle_t - 2d\langle W^1, W^2\rangle_t\big) \\
&\quad + 2\epsilon\sigma \mathrm{e}^{\kappa^* t}\big(V_t^1 - V_t^2\big)\big(dB_t^{V^1} - dB_t^{V^2}\big) + 2\sqrt{\epsilon}\sigma \mathrm{e}^{\kappa^* t}\big(W_t^1 - W_t^2\big)\big(dB_t^{W^1} - dB_t^{W^2}\big).
\end{aligned}$$

Of course, this time the covariation of V^1 and V^2 as well as W^1 and W^2 are 0 due to independence and we have a positive Itô-correction. This implies the inequality

$$\begin{aligned}
\mathrm{e}^{\kappa^* t} V\big(V_t^1, W_t^1, V_t^2, W_t^2\big) \le V\big(V_0^1, W_0^1, V_0^2, W_0^2\big) &+ 4\epsilon\sigma^2 \int_0^t \mathrm{e}^{\kappa^* s}\, ds \\
&+ 2\epsilon\sigma \int_0^t \mathrm{e}^{\kappa^* s}\big(V_s^1 - V_s^2\big)\big(dB_s^{V^1} - dB_s^{V^2}\big) \\
&+ 2\sqrt{\epsilon}\sigma \int_0^t \mathrm{e}^{\kappa^* s}\big(W_s^1 - W_s^2\big)\big(dB_s^{W^1} - dB_s^{W^2}\big)
\end{aligned}$$

or equivalently

$$\begin{aligned}
V\big(V_t^1, W_t^1, V_t^2, W_t^2\big) \le \mathrm{e}^{-\kappa^* t} V\big(V_0^1, W_0^1, V_0^2, W_0^2\big) &+ 4\epsilon\sigma^2 \int_0^t \mathrm{e}^{\kappa^*(s-t)}\, ds \\
&+ \mathrm{e}^{-\kappa^* t} M_t^V + \mathrm{e}^{-\kappa^* t} M_t^W,
\end{aligned}$$

where the stochastic integrals are denoted by M_t^V and M_t^W, respectively. Taking expectations—using $\mathbb{E}\left[M_t^V\right] = \mathbb{E}\left[M_t^W\right] = 0$—we obtain the estimate

$$\mathbb{E}\left[V\big(V_t^1, W_t^1, V_t^2, W_t^2\big)\right] \le \mathrm{e}^{-\kappa^* t} V\big(V_0^1, W_0^1, V_0^2, W_0^2\big) + \frac{4\epsilon\sigma^2}{\kappa^*}. \qquad \square$$

EXAMPLE 6.8. $a = 0.25$, $\epsilon = 0.01$, $\gamma = 0.95 \cdot \gamma^*(a)$ and $I = 0.06$ as before. For $\theta = 1.1 \cdot \theta^*$, i. e. Proposition 6.6 holds, we observe the following behavior depending on the noise intensity σ.

6.4. Coupled Andronov-Hopf Oscillators

We already got to know the so-called Andronov-Hopf oscillator in Section 5.4 and in this section we study two of them coupled linearly. However, let us first comment on the neurobiological relevance of this particular example. Consider a nonlinear two-dimensional system of ODEs

$$\frac{d}{dt}\begin{bmatrix} X_t \\ Y_t \end{bmatrix} = \begin{bmatrix} f\big(X_t, Y_t, \mu\big) \\ g\big(X_t, Y_t, \mu\big) \end{bmatrix},$$

where μ is a parameter. Suppose that $(0,0)$ is an equilibrium point and the linearized system has eigenvalues $\lambda_\pm = \mu \pm \mathrm{i}\omega$, $\omega > 0$. We know that the equilibrium is stable if $\mu < 0$ but here we are interested in what happens as μ changes its sign, i. e. the complex conjugate pair of eigenvalues of the linearized flow crosses the imaginary axis and is purely imaginary for $\mu = 0$. Such a situation is known as a

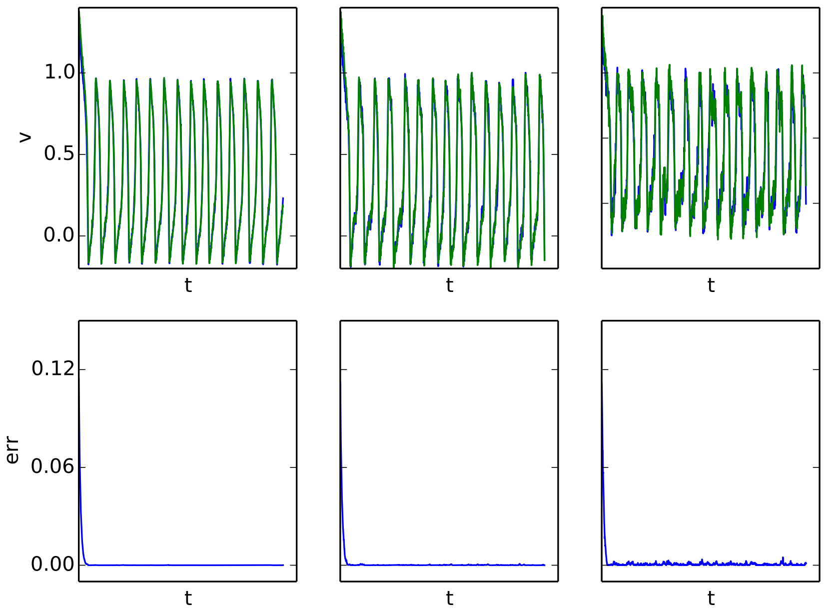

FIGURE 3. A plot of V_t^1 and V_t^2 in blue and green, as well as $\text{err} = \epsilon(V_t^1 - V_t^2)^2 + (W_t^1 - W_t^2)^2$ for three different values $\sigma = 0.01$, $\sigma = 0.02$ and $\sigma = 0.05$ from left to right.

Hopf bifurcation which is the simplest form of a bifurcation that allows for emerging periodic solutions from an equilibrium as a parameter crosses a critical value.

It is intuitively clear, that there appears a *limit cycle* in such a situation. The equilibrium looses stability and the orbits are spiralling out as $\mu > 0$. However, the linearization is only valid in a small neighborhood of 0, thus this change in stability is only a local change and the flow far from the equilibrium remains qualitatively unaffected. If we assume that the the nonlinearity is contracting outside a certain region, a limit cycle should appear where these forces balance with the local instability.

Such dynamical systems at a Hopf bifurcation can be transformed into their *Hopf normal form* which involves a number of coordinate changes and in general results in a system

$$\frac{d}{dt} Z_t = (\mu + \mathrm{i}\omega) Z_t - (a + \mathrm{i}b)|Z_t|^2 Z_t + O\big(Z_t^5\big), \tag{6.4}$$

where $z = x + \mathrm{i}y$. The key point of this approximation is to find the sign of a, which decides wether the bifurcation is *subcritical* ($a > 0$) or *supercritical* ($a < 0$), that is the limit cycle is unstable and stable, respectively. There is an explicit expression for a in terms of derivates of f and g up to third order but the derivation of such a formula is beyond the scope of this lecture.

Now, setting $\mu = \omega = a = 1$ and $b = 0$ together with the transformation to $z = (x, y) \in \mathbb{R}^2$ (6.4) is up to higher order terms equivalent to the Andronov-Hopf oscillator

$$\frac{d}{dt} Z_t = b\big(Z_t\big), \tag{6.5}$$

where

$$b(z) := \begin{bmatrix} x - y - x\|z\|^2 \\ y + x - y\|z\|^2 \end{bmatrix} = z\big(1 + \|z\|^2\big) + z^{\perp}.$$

We used the notation $z^{\perp} = (-y, x)$ for the vector orthogonal to z. Summing up, (6.5) is the prototype for the local description of dynamical systems that have a limit cycle due to a Hopf bifurcation and thus can be seen as a prototype of a neural oscillator. The following proposition concerns the transformation to polar coordinates.

PROPOSITION 6.9. *For $z \in \mathbb{R}^2$ we can write $z = \|z\| \frac{z}{\|z\|} = r \cdot n$ where r denotes the radial part and n the angular part. Thus, for all $Z_0 \neq 0 \in \mathbb{R}^2$ we have $Z_t = r_t n_t$ with*

$$\frac{d}{dt} n_t = \begin{bmatrix} -n_t^2 \\ n_t^1 \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix} n_t, \quad \textit{thus} \quad n_t = \mathrm{e}^{\mathrm{i}t} n_0,$$

i. e. the angular part describes a rotation. Moreover

$$\frac{d}{dt} r_t = r_t\big(1 - r_t^2\big), \quad \textit{hence} \quad r_t = \frac{\|Z_0\|}{\sqrt{\|Z_0\|^2 + \big(1 - \|Z_0\|^2\big)\mathrm{e}^{-2t}}} \xrightarrow{t \to \infty} 1.$$

Now consider two linearly coupled AH-oscillators

$$\begin{aligned} dZ_t^1 &= \Big(b\big(Z_t^1\big) + \theta\big(Z_t^2 - Z_t^1\big)\Big)\, dt + \sigma\, dW_t, \\ dZ_t^2 &= \Big(b\big(Z_t^2\big) + \theta\big(Z_t^1 - Z_t^2\big)\Big)\, dt + \sigma\, dW_t, \end{aligned} \tag{6.6}$$

where $\theta \geq 0$ denotes again the coupling strength, $\sigma \geq 0$ and W is a common two-dimensional Brownian motion. For large enough coupling strength the asymptotic synchronization can be proven as a corollary to Proposition 6.4, since

$$\begin{aligned} \langle b(z^1) - b(z^2), z^1 - z^2 \rangle &= \langle z^1, z^1 - z^2 \rangle \big(1 - \|z^1\|^2\big) - \langle z^2, z^1 - z^2 \rangle \big(1 - \|z^2\|^2\big) \\ &= \|z^1 - z^2\|^2 \Big(1 - \frac{1}{2}\big(\|z^1\|^2 + \|z^2\|^2\big)\Big) - \frac{1}{2}\big(\|z^1\|^2 - \|z^2\|^2\big)^2 \\ &\leq \|z^1 - z^2\|^2. \end{aligned}$$

COROLLARY 6.10. *Let $\theta > \frac{1}{2}$, hence $\kappa^* := 2(2\theta - 1) > 0$. Then for all ω*

$$\|Z_t^1 - Z_t^2\|^2(\omega) \leq \mathrm{e}^{-\kappa^* t} \|Z_0^1 - Z_0^2\|^2(\omega)$$

In particular, the system (6.6) *is synchronizing pathwise and the synchronized state is* pathwise exponentially attracting *with* exponential rate κ^*.

Of course, the nonlinearity b is one-sided Lipschitz but this estimate is rather crude and we neglected a couple of negative terms. In the following we analyze this in more detail and obtain a result on synchronization for arbitrary small coupling strength in the deterministic case. Thus, in the following set $\sigma = 0$.

LEMMA 6.11. *Let $\|Z_0^1\|^2 + \|Z_0^2\|^2 > 0$. Then for all $t \geq 0$ it holds that*

$$\frac{2\mathrm{e}^{2t}}{\mathrm{e}^{2t}-1} \geq \|Z_t^1\|^2 + \|Z_t^2\|^2 \geq \begin{cases} \frac{\mathrm{e}^{2(1-2\theta)t}\left(\|Z_0^1\|^2+\|Z_0^2\|^2\right)}{1+\frac{1}{1-2\theta}\left(\mathrm{e}^{2(1-2\theta)t}-1\right)\left(\|Z_0^1\|^2+\|Z_0^2\|^2\right)} > 0 & \text{if } \theta \neq \frac{1}{2} \text{ and} \\ \frac{\left(\|Z_0^1\|^2+\|Z_0^2\|^2\right)}{1+2t\left(\|Z_0^1\|^2+\|Z_0^2\|^2\right)} > 0 & \text{if } \theta = \frac{1}{2}. \end{cases}$$

PROOF. Indeed it is immediate to check that

$$\begin{aligned} \frac{1}{2}\frac{d}{dt}\left(\|Z_t^1\|^2 + \|Z_t^2\|^2\right) &= \langle b(Z_t^1), Z_t^1\rangle + \langle b(Z_t^2), Z_t^2\rangle - \theta\|Z_t^1 - Z_t^2\|^2 \\ &= \left(\|Z_t^1\|^2 + \|Z_t^2\|^2\right) - \left(\|Z_t^1\|^4 + \|Z_t^2\|^4\right) - \theta\|Z_t^1 - Z_t^2\|^2 \\ &\geq (1-2\theta)\left(\|Z_t^1\|^2 + \|Z_t^2\|^2\right) - \left(\|Z_t^1\|^2 + \|Z_t^2\|^2\right)^2 \end{aligned}$$

and on the other hand

$$\frac{1}{2}\frac{d}{dt}\left(\|Z_t^1\|^2 + \|Z_t^2\|^2\right) \leq \|Z_t^1\|^2 + \|Z_t^2\|^2 - \frac{1}{2}\left(\|Z_t^1\|^2 + \|Z_t^2\|^2\right)^2.$$

Thus, consider the differential inequalities $2g_t - g_t^2 \geq \frac{d}{dt}g_t \geq 2(1-2\theta)g_t - 2g_t^2$ whose solution is

$$\frac{\mathrm{e}^{2t}g_0}{1+\frac{1}{2}\left(\mathrm{e}^{2t}-1\right)g_0} \geq g_t \geq \begin{cases} \frac{\mathrm{e}^{2(1-2\theta)t}g_0}{1+\frac{1}{1-2\theta}\left(\mathrm{e}^{2(1-2\theta)t}-1\right)g_0} > 0 & \text{if } \theta \neq \frac{1}{2} \text{ and} \\ \frac{g_0}{1+2tg_0} > 0 & \text{if } \theta = \frac{1}{2}. \end{cases}$$ □

LEMMA 6.12. *Let $Z_0^1, Z_0^2 \neq 0$, $T_* = \inf\{t \geq 0 : Z_t^1 = 0 \text{ or } Z_t^2 = 0\} > 0$ by continuity with $\inf\emptyset = +\infty$. Furthermore, define the angle between both vectors as*

$$g_t = \frac{\langle Z_t^1, Z_t^2\rangle}{\|Z_t^1\|\|Z_t^2\|}, \quad t < T_*.$$

Then

$$g_t = \frac{e^{h_t} - e^{-h_t}}{e^{h_t} + e^{-h_t}} = \tanh\left(h_t\right),$$

where

$$h_t = \operatorname{arctanh}\left(g_0\right) + \theta\int_0^t \left(\frac{\|Z_s^1\|}{\|Z_s^2\|} + \frac{\|Z_s^2\|}{\|Z_s^1\|}\right) ds \geq \operatorname{arctanh}\left(g_0\right) + 2\theta t.$$

In particular, $\lim_{t\nearrow T_*} g_t = 1$.

PROOF. **Step 1:**

$$\begin{aligned} \frac{d}{dt}\langle Z_t^1, Z_t^2\rangle &= \langle \dot{Z}_t^1, Z_t^2\rangle + \langle Z_t^1, \dot{Z}_t^2\rangle \\ &= \langle Z_t^1, Z_t^2\rangle + \langle \left(Z_t^1\right)^\perp, Z_t^2\rangle - \langle Z_t^1, Z_t^2\rangle\|Z_t^1\|^2 + \theta\langle Z_t^2 - Z_t^1, Z_t^2\rangle \\ &\quad + \langle Z_t^1, Z_t^2\rangle + \langle Z_t^1, \left(Z_t^2\right)^\perp\rangle - \langle Z_t^1, Z_t^2\rangle\|Z_t^2\|^2 + \theta\langle Z_t^1 - Z_t^2, Z_t^1\rangle \\ &= \left(2 - \|Z_t^1\|^2 - \|Z_t^2\|^2\right)\langle Z_t^1, Z_t^2\rangle + \theta\|Z_t^1 - Z_t^2\|^2, \end{aligned}$$

where, for the last equality, we used

$$\langle \left(z^1\right)^\perp, z^2\rangle + \langle z^1, \left(z^2\right)^\perp\rangle = -y^1x^2 + x^1y^2 + x^1(-y^2) + y^1x^2 = 0.$$

Step 2:

$$\begin{aligned}\frac{d}{dt}\frac{1}{\|Z_t^1\|\|Z_t^2\|} &= -\frac{\langle \dot Z_t^1, Z_t^1\rangle}{\|Z_t^1\|^3\|Z_t^2\|} - \frac{\langle \dot Z_t^2, Z_t^2\rangle}{\|Z_t^1\|\|Z_t^2\|^3}\\ &= -\frac{\|Z_t^1\|^4 - \|Z_t^1\|^2 - \theta\langle Z_t^1, Z_t^2 - Z_t^1\rangle}{\|Z_t^1\|^3\|Z_t^2\|} + \frac{\|Z_t^2\|^4 - \|Z_t^2\|^2 - \theta\langle Z_t^2, Z_t^1 - Z_t^2\rangle}{\|Z_t^1\|\|Z_t^2\|^3}\\ &= -\frac{\big(2 - \|Z_t^1\|^2 - \|Z_t^2\|^2\big)}{\|Z_t^1\|\|Z_t^2\|} + \frac{2\theta}{\|Z_t^1\|\|Z_t^2\|} - \theta\frac{\langle Z_t^1, Z_t^2\rangle}{\|Z_t^1\|\|Z_t^2\|}\left(\frac{1}{\|Z_t^1\|^2} + \frac{1}{\|Z_t^2\|^2}\right)\end{aligned}$$

Hence, we can combine both steps to

$$\begin{aligned}\frac{d}{dt}g_t &= \big(2 - \|Z_t^1\|^2 - \|Z_t^2\|^2\big)g_t + \theta\frac{\|Z_t^1 - Z_t^2\|^2}{\|Z_t^1\|\|Z_t^2\|}\\ &\quad - \big(2 - \|Z_t^1\|^2 - \|Z_t^2\|^2\big)g_t + 2\theta g(t) - \theta g_t^2\left(\frac{\|Z_t^1\|}{\|Z_t^2\|} + \frac{\|Z_t^2\|}{\|Z_t^1\|}\right)\\ &= \theta\left(\frac{\|Z_t^1\|}{\|Z_t^2\|} + \frac{\|Z_t^2\|}{\|Z_t^1\|}\right)\big(1 - g_t^2\big) =: f_t\big(1 - g_t^2\big).\end{aligned}$$

The ordinary differential equation $\frac{d}{dt}g_t = f_t(1 - g_t^2)$ can be integrated as follows.

$$\int_0^t \frac{g_s'}{1 - g_s^2}\,ds = \int_0^t f_s\,ds \quad\Leftrightarrow\quad \operatorname{arctanh}\big(g_t\big) - \operatorname{arctanh}\big(g_0\big) = \int_0^t f_s\,ds.$$

Obviously, f is bounded from below by 2, thus in the case $T_* = +\infty$ it follows $\lim_{t\nearrow T_*} g_t = 1$. Now assume $T_* < +\infty$ and w. l. o. g. $Z_{T_*}^1 = 0$. But then $\lim_{t\uparrow T_*}\|Z_t^2\| > 0$ due to Lemma 6.11. Since

$$Z_{T_*}^1 = Z_t^1 + \int_t^{T_*} b(Z_s^1) + \theta(Z_s^2 - Z_s^1)\,ds$$

it follows that $\|Z_t^1\| \le C(T_* - t)$ with constant $C := \max_{0\le s\le T_*}\|b(Z_s^1) + \theta(Z_s^2 - Z_s^1)\| < \infty$. Hence, $f_s \ge \theta\frac{\|Z_t^2\|}{C(T_*-t)}$ is no longer integrable on $[0, T_*]$ which is a contradiction and it follows that $\lim_{t\nearrow T_*} g_t = 1$. □

Now we are in the position to prove the following result on synchronization for arbitrary coupling strength $\theta > 0$.

Proposition 6.13 (Deterministic Case). *Let $Z_0^1, Z_0^2 \ne 0 \in \mathbb{R}^2$. Then there exists some $t_0 > 0$ and some constant $C = C(Z_0^1, Z_0^2, t_0)$ such that $\langle Z_{t_0}^1, Z_{t_0}^2\rangle > 0$ and*

$$\|Z_t^1 - Z_t^2\|^2 \le C\mathrm{e}^{-4\theta t}\|Z_0^1 - Z_0^2\|^2, \quad t \ge t_0$$

The explicit constant C is given by

$$C = \exp\left(\int_0^{t_0} 2 - \|Z_s^1\|^2 - \|Z_s^2\|^2\,ds\right)\frac{\mathrm{e}^{2t_0}}{\mathrm{e}^{2t_0} - 1}\frac{1}{\langle Z_{t_0}^1, Z_{t_0}^2\rangle}.$$

In other words, the synchronized state is exponentially attracting with rate 4θ once the phase difference between the two oscillators is below $90°$. This happens in finite time t_0.

PROOF. By Lemma 6.12, $\lim_{t\nearrow T_*} g_t = 1$, thus there exists $t_0 > 0$ such that $g_{t_0} > 0$ regardless of the initial conditions, as long as $Z_0^1, Z_0^2 \neq 0$. Hence $\langle Z_{t_0}^1, Z_{t_0}^2\rangle > 0$ and Step 1 of Lemma 6.12 implies

$$\langle Z_t^1, Z_t^2\rangle \geq \exp\left(\int_{t_0}^t \left(2 - \|Z_s^1\|^2 - \|Z_s^2\|^2\right)\, ds\right)\langle Z_{t_0}^1, Z_{t_0}^2\rangle > 0,$$

hence

$$\int_{t_0}^t \left(2 - \|Z_s^1\|^2 - \|Z_s^2\|^2\right) ds \leq \log\left(\frac{\langle Z_t^1, Z_t^2\rangle}{\langle Z_{t_0}^1, Z_{t_0}^2\rangle}\right) \leq \log\left(\frac{\|Z_t^1\|^2 + \|Z_t^2\|^2}{2\langle Z_{t_0}^1, Z_{t_0}^2\rangle}\right)$$

which can be bounded uniformly in $t \geq t_0$ by Lemma 6.11. This implies

$$\begin{aligned}
\frac{d}{dt}\|Z_t^1 - Z_t^2\|^2 &= (2-4\theta)\|Z_t^1 - Z_t^2\|^2 - 2\|Z_t^1\|^2\langle Z_t^1, Z_t^1 - Z_t^2\rangle + 2\|Z_t^2\|^2\langle Z_t^2, Z_t^1 - Z_t^2\rangle \\
&= (2-4\theta)\|Z_t^1 - Z_t^2\|^2 - 2\|Z_t^1\|^4 - 2\|Z_t^2\|^4 + 2\langle Z_t^1, Z_t^2\rangle\left(\|Z_t^1\|^2 + \|Z_t^2\|^2\right) \\
&= (2-4\theta)\|Z_t^1 - Z_t^2\|^2 - \left(\|Z_t^1\|^2 - \|Z_t^2\|^2\right)^2 \\
&\quad - \left(\|Z_t^1\|^2 + \|Z_t^2\|^2\right)\left(\|Z_t^1\|^2 + \|Z_t^2\|^2 - 2\langle Z_t^1, Z_t^2\rangle\right) \\
&\leq \left(2 - \|Z_t^1\|^2 - \|Z_t^2\|^2 - 4\theta\right)\|Z_t^1 - Z_t^2\|^2 .
\end{aligned}$$

Consequently,

$$\begin{aligned}
\|Z_t^1 - Z_t^2\|^2 &\leq \exp\left(\int_0^t 2 - \|Z_s^1\|^2 - \|Z_s^2\|^2\, ds - 4\theta t\right)\|Z_0^1 - Z_0^2\|^2 \\
&= \exp\left(\int_0^{t_0} 2 - \|Z_s^1\|^2 - \|Z_s^2\|^2 ds\right)\frac{\|Z_t^1\|^2 + \|Z_t^2\|^2}{2\langle Z_{t_0}^1, Z_{t_0}^2\rangle}\mathrm{e}^{-4\theta t}\|Z_0^1 - Z_0^2\|^2.
\end{aligned}$$

Now use the upper bound from Lemma 6.11 to obtain the result. □

6.5. Coupling of Neurons Through Noise

In the last section we had some trouble proving a local synchronization result for the system of two linearly coupled Andronov-Hopf oscillators. However, the generalization to a noisy system is unclear. In this section we want to introduce a simple framework, where the noise actually improves the synchronization. Consider again

$$\begin{aligned}
dZ_t^1 &= \left(b\left(Z_t^1\right) + \theta\left(Z_t^2 - Z_t^1\right)\right) dt + \sigma Z_t^1\, dW_t \\
dZ_t^2 &= \left(b\left(Z_t^2\right) + \theta\left(Z_t^1 - Z_t^2\right)\right) dt + \sigma Z_t^2\, dW_t
\end{aligned} \tag{6.7}$$

where W is a one-dimensional Brownian motion. Compared to (6.6) we use multiplicative noise with a common Brownian motion that implies an additional coupling of the two oscillators. Indeed, we can improve Corollary 6.10 to

PROPOSITION 6.14 (Global Synchronization). *Let $Z_0^1 \neq Z_0^2 \in \mathbb{R}^2$. Then*

$$\|Z_t^1 - Z_t^2\|^2 \leq \exp\left(\left(2 - 4\theta - \sigma^2\right)t + 2\sigma W_t\right)\|Z_0^1 - Z_0^2\|^2$$

In particular, if $\sigma^2 + 4\theta > 2$, then

$$\lim_{t\to\infty}\|Z_t^1 - Z_t^2\| = 0 \quad \mathbb{P}\text{-a. s.}$$

with exponential rate κ^* *for any* $\kappa^* < \sigma^2 + 4\theta - 2$.

PROOF. Fix $\epsilon > 0$. Then Itô's formula implies

$$\begin{aligned} d\log\big(\epsilon + \|Z_t^1 - Z_t^2\|^2\big) = & \frac{2\langle b(Z_t^1) - b(Z_t^2), Z_t^1 - Z_t^2\rangle - 4\theta\|Z_t^1 - Z_t^2\|^2}{\epsilon + \|Z_t^1 - Z_t^2\|^2}\,dt \\ & + \frac{2\sigma\langle Z_t^1, Z_t^1 - Z_t^2\rangle - 2\sigma\langle Z_t^2, Z_t^1 - Z_t^2\rangle}{\epsilon + \|Z_t^1 - Z_t^2\|^2}\,dW_t \\ & + \text{ Itô-correction terms} \end{aligned}$$

To calculate the Itô-correction terms note that:

$$\partial_{x^1x^1}\log\big(\epsilon + \|z^1 - z^2\|^2\big) = \partial_{x^1}\frac{2(x^1 - x^2)}{\epsilon + \|z^1 - z^2\|^2} = \frac{2}{\epsilon + \|z^1 - z^2\|^2} - \frac{4(x^1 - x^2)^2}{\big(\epsilon + \|z^1 - z^2\|^2\big)^2}$$

$$\partial_{x^2x^2}\log\big(\epsilon + \|z^1 - z^2\|^2\big) = \frac{2}{\epsilon + \|z^1 - z^2\|^2} - \frac{4(x^1 - x^2)^2}{\big(\epsilon + \|z^1 - z^2\|^2\big)^2}$$

$$\partial_{x^1x^2}\log\big(\epsilon + \|z^1 - z^2\|^2\big) = -\frac{2}{\epsilon + \|z^1 - z^2\|^2} + \frac{4(x^1 - x^2)^2}{\big(\epsilon + \|z^1 - z^2\|^2\big)^2}$$

$$\partial_{x^1y^1}\log\big(\epsilon + \|z^1 - z^2\|^2\big) = -\frac{4(x^1 - x^2)(y^1 - y^2)}{\big(\epsilon + \|z^1 - z^2\|^2\big)^2} = \partial_{x^2y^2}\log\big(\epsilon + \|z^1 - z^2\|^2\big)$$

$$\partial_{x^1y^2}\log\big(\epsilon + \|z^1 - z^2\|^2\big) = \frac{4(x^1 - x^2)(y^1 - y^2)}{\big(\epsilon + \|z^1 - z^2\|^2\big)^2} = \partial_{y^1x^2}\log\big(\epsilon + \|z^1 - z^2\|^2\big).$$

and similar expressions for the remaining derivates. Also, the covariations are easy to obtain. Essentially these are $\langle X^iW, X^jW\rangle_t = X_t^iX_t^jt$, $\langle X^iW, Y^jW\rangle_t = X_t^iY_t^jt$ and $\langle Y^iW, Y^jW\rangle_t = Y_t^iY_t^jt$. It is immediate to see that one arrives at expressions of the type

$$\langle X^1W, X^1W\rangle_t + \langle X^2W, X^2W\rangle_t - \langle X^1W, X^2W\rangle_t - \langle X^2W, X^1W\rangle_t = \big(X_t^1 - X_t^2\big)^2t$$

and the same for the Y-variables. Thus, the correction term reads as

$$\begin{aligned} = & \frac{\sigma^2\big(X_t^1 - X_t^2\big)^2 + \sigma^2\big(Y_t^1 - Y_t^2\big)^2}{\epsilon + \|Z_t^1 - Z_t^2\|^2}\,dt \\ & - \frac{2\sigma^2\Big(\big(X_t^1 - X_t^2\big)^4 + \big(Y_t^1 - Y_t^2\big)^4 + 2\big(X_t^1 - X_t^2\big)^2\big(Y_t^1 - Y_t^2\big)^2\Big)}{\big(\epsilon + \|Z_t^1 - Z_t^2\|^2\big)^2}\,dt \\ = & \frac{\sigma^2\|Z_t^1 - Z_t^2\|^2}{\epsilon + \|Z_t^1 - Z_t^2\|^2}\,dt - \frac{2\sigma^2\|Z_t^1 - Z_t^2\|^4}{\big(\epsilon + \|Z_t^1 - Z_t^2\|^2\big)^2}\,dt. \end{aligned}$$

In integral form we have shown the following using the one-sided Lipschitz condition on b (with constant $L = 1$)

$$\begin{aligned} \log\big(\epsilon + \|Z_t^1 - Z_t^2\|^2\big) \le \log\big(\epsilon + \|Z_0^1 - Z_0^2\|^2\big) + (2 - 4\theta)\int_0^t \frac{\|Z_s^1 - Z_s^2\|^2}{\epsilon + \|Z_s^1 - Z_s^2\|^2}\,ds \\ + 2\sigma\int_0^t \frac{\|Z_s^1 - Z_s^2\|^2}{\epsilon + \|Z_s^1 - Z_s^2\|^2}\,dW_s \\ + \sigma^2\int_0^t \frac{\|Z_s^1 - Z_s^2\|^2}{\epsilon + \|Z_s^1 - Z_s^2\|^2}\,ds - 2\sigma^2\int_0^t \frac{\|Z_s^1 - Z_s^2\|^4}{\big(\epsilon + \|Z_s^1 - Z_s^2\|^2\big)^2}\,ds. \end{aligned}$$

Since the exponential function is monotone, we can take the exponential on both sides and then take the limit $\epsilon \searrow 0$ to obtain the first assertion.

If in addition $\sigma^2 + 4\theta > 2$ then

$$\big(2 - 4\theta - \sigma^2\big)t + 2\sigma W_t = \left(2 - 4\theta - \sigma^2 + 2\sigma \frac{W_t}{t}\right) t \to -\infty \quad \mathbb{P}\text{-a.s.}$$

because $\frac{W_t}{t} \to 0$ $\mathbb{P}$-a.s. by the strong law of large numbers for the Brownian motion. □

Remark 6.15. (1) Proposition 6.14 demonstrates the effect of noise on synchronization. In particular, common noise can also couple oscillators even for negative coupling constant θ!

(2) Furthermore, Proposition 6.14 remains true for any system of coupled oscillators as soon as b satisfies the one-sided Lipschitz condition

$$\langle b\big(z^1\big) - b\big(z^2\big), z^1 - z^2 \rangle \le L \|z^1 - z^2\|^2.$$

Then, the criterion is $\sigma^2 + 4\theta > 2L$ and the exponential rate is $\kappa^* < \sigma^2 + 4\theta - 2L$.

This is the case for the FitzHugh-Nagumo system, where $z = (v, w)$ and

$$b(v,w) = \begin{bmatrix} f(v) - w + I \\ \epsilon(v - \gamma w) \end{bmatrix}, \quad f(v) = v(1-v)(v-a), \quad \epsilon, \gamma > 0.$$

Here, we can calculate

$$\begin{aligned} &\langle b\big(v^1, w^1\big) - b\big(v^2, w^2\big), \begin{bmatrix} v^1 \\ w^1 \end{bmatrix} - \begin{bmatrix} v^2 \\ w^2 \end{bmatrix} \rangle \\ &= \Big(f\big(v^1\big) - f\big(v^2\big)\Big)\big(v^1 - v^2\big) + (\epsilon - 1)\big(v^1 - v^2\big)\big(w^1 - w^2\big) - \epsilon\gamma\big(w^1 - w^2\big)^2 \\ &\le \left(\frac{1 - a + a^2}{3} + \frac{(\epsilon - 1)^2}{2}\right)\big(v^1 - v^2\big)^2 + \left(\frac{1}{2} - \epsilon\gamma\right)\big(w^1 - w^2\big)^2 \le L\|z^1 - z^2\|^2, \end{aligned}$$

where $L = \max\{\frac{1-a+a^2}{3} + \frac{(\epsilon-1)^2}{2}, \frac{1}{2} - \epsilon\gamma\}$. To obtain the second to last equality note that $\sup_{\xi \in \mathbb{R}} f'(\xi) = \frac{1-a+a^2}{3}$.

(3) Proposition 6.14 also remains true if we add independent additive noise to the coupled system, i.e. for systems of the type

$$\begin{aligned} dZ_t^1 &= \Big(b\big(Z_t^1\big) + \theta\big(Z_t^2 - Z_t^1\big)\Big)\, dt + \sigma Z_t^1 \, dW_t + \hat{C}\, d\hat{W}_t \\ dZ_t^2 &= \Big(b\big(Z_t^2\big) + \theta\big(Z_t^1 - Z_t^2\big)\Big)\, dt + \sigma Z_t^2 \, dW_t + \hat{C}\, d\hat{W}_t \end{aligned}$$

where $\hat{C} \in \mathbb{R}^{2\times d}$ and $\hat{W}$ is a d-dimensional Brownian motion, $d \ge 1$, independent of W. Indeed, the calculations for the proof are the same up to the calculations of the Itô-correction term since in the difference the noise vanishes and also the additional contribution to the covariation between the components of Z_t^1, Z_t^2 vanishes as observed before.

CHAPTER 7

Systems of Coupled Neurons

In this chapter, we consider a large network of N neurons, instead of a system of two neurons only as done before. The number N can vary among different magnitudes depending on purpose and location of the network in the brain. As in Chapter 6 we simplify the synaptic connections between the neurons and neglect any dynamical behavior, which would exclude e. g. learning or short term memory.

In the following, we consider networks of neurons of the same type modelled as before with stochastic differential equations

$$dZ_t^i = \left(b\big(\alpha_i, Z_t^i\big) + \sum_{j=1}^{N} \Gamma_{ij}\big(Z_t^j - Z_t^i\big) \right) dt + \sigma Z_t^i \, dW_t + C \, dB_t + \tilde{\sigma} \, dB_t^i, \tag{7.1}$$

where $Z_t^{\in}\mathbb{R}^d$, $d \geq 1$, α_i is a parameter for the dynamics of the single neuron, introducing so-called disorder, $\Gamma_{ij} : \mathbb{R}^d \to \mathbb{R}^d$ is the pairwise interaction function, $C \in \mathbb{R}^{d \times m}$, $m \geq 1$, $\sigma, \tilde{\sigma} > 0$ and W, B, B^i are independent one-, m- and d-dimensional Brownian motions, respectively.

Of special interest are networks of limit-cycle oscillators, where $Z^i \in \mathbb{R}$ describes the phase of the ith oscillator. The most prominent example in this direction is the Kuramoto model studied in Section 7.1. Also, we consider a simplified version of our usual suspect, the FitzHugh-Nagumo system in Section 7.4. In these to examples the two extreme choices for the connections of our network are used, namely *all-to-all* or *mean-field coupling* and *nearest-neighbor coupling*.

(1) *All-to-all coupling:* In this case the choice of the connection function is $\Gamma_{ij} = \frac{\theta}{N}\Gamma$, $\theta > 0$, $\Gamma : \mathbb{R}^d \to \mathbb{R}^d$. Thus every pair of oscillators is coupled with the same coupling strength and the same connection function. Note that the coupling becomes small if N is large. That is why this sort of coupling is usually referred to as *weak coupling*.

(2) *Nearest-neighbor coupling:* Think of the oscillators arranged on a line or on a circle. For given $i \in \{1, \ldots, N\}$ denote by $N(i)$ the set of nearest neighbors, i. e. $N(1) = \{2\}$, $N(N) = \{N-1\}$ and $N(i) = \{i-1, i+1\}$, $i = 2, \ldots, N-1$ on the line as well as $N(1) = \{2, N\}$, $N(N) = \{1, N-1\}$ and $N(i) = \{i-1, i+1\}$, $i = 2, \ldots, N-1$ on the circle. Now set $\Gamma_{ij} \neq 0$ if and only if $j \in N(i)$.

7.1. The Kuramoto Model

The Kuramoto model is a rather simple mathematical model that allows to study synchronization in neural ensembles. It is based on the following universal form for the long-term dynamics of coupled, nearly identical limit-cycle oscillators, given by stochastic differential equations for the phase θ_i of each oscillator.

$$d\theta_i(t) = \left(\omega_i + \sum_{j=1}^{N} \Gamma_{ij}\big(\theta_j(t) - \theta_i(t)\big)\right) dt + \sigma\, dW_t^i.$$

Here, ω_i is the intrinsic frequency of the ith oscillator. Now assuming all-to-all coupling and a 2π-periodic coupling function Γ, one can expand it into a Fourier series and keeping only the first term one obtains

$$d\theta_i(t) = \left(\omega_i + \frac{K}{N}\sum_{j=1}^{N} \sin\big(\theta_j(t) - \theta_i(t)\big)\right) dt + \sigma\, dW_t^i.$$

The case $\sigma = 0$ is the original system studied by Kuramoto, see e. g. [**ET10, Str00, Kur84**]. The frequencies are distributed according to some symmetric and unimodal probability density $g(\omega)$ where the mean can be set to 0 using a rotating frame of reference. This leaves the equations invariant due to the rotational symmetry in the system.

One distinctive feature of the all-to-all coupling in this case is that—imagine each oscillator represents a complex number on the unit circle—the mean introduces the so-called order parameter

$$r(t)\mathrm{e}^{\mathrm{i}\psi(t)} = \frac{1}{N}\sum_{j=1}^{N} \mathrm{e}^{\mathrm{i}\theta_j(t)}.$$

This quantity can be seen as a collective, macroscopic rhythm induced by the population of oscillators, where the radius r represents the phase coherence and ψ the average phase. Multiplying both sides with $\mathrm{e}^{\mathrm{i}\theta_i(t)}$ we obtain

$$r(t)\mathrm{e}^{\mathrm{i}\left(\psi(t)-\theta_i(t)\right)} = \frac{1}{N}\sum_{j=1}^{N} \mathrm{e}^{\mathrm{i}\left(\theta_j(t)-\theta_i(t)\right)}$$

hence for the imaginary part

$$r(t)\sin\big(\psi(t) - \theta_i(t)\big) = \frac{1}{N}\sum_{j=1}^{N} \sin\big(\theta_j(t) - \theta_i(t)\big), \quad i = 1, \ldots, N.$$

Inserting this back into the original equation for $\dot\theta_k$ we arrive at

$$\dot\theta_i(t) = \omega_i + Kr(t)\sin\big(\psi(t) - \theta_i(t)\big).$$

In this form, the naming *mean-field coupling* becomes obvious, since each oscillator is coupled to the mean phase ψ with coupling constant $K \cdot r$ and there is no other dependence on the remaining oscillators. A quantity $r \approx 1$ corresponds to a cluster of synchronized oscillators behaving a one single oscillator, whereas at $r \approx 0$ each oscillator acts independently without a collective rhythm.

Using numerical integration, we can observe the time evolution of $r(t)$. For this purpose, we fix $g(\omega)\, d\omega$ to be a standard normal distribution and vary the coupling constant K. We have already seen in Chapter 6 that with $K \gg 1$ it is more likely to obtain an asymptotically synchronized state. Also for small K such a system may not always synchronize. In the Kuramoto model there appears to be a critical value K_c such that below this threshold $r(t) \to 0$ with $O(N^{-\frac{1}{2}})$ fluctuations—as it is natural for N uncoupled oscillators. The system converges to the so-called *incoherent state*. Above this critical value the order parameter rises exponentially and $r(t) \to r_\infty < 1$, again with the same size of fluctuations. In this case, the oscillators split into two groups. One is a synchronized cluster and the second one is a bunch of oscillators that rotate at their own, large enough frequency. This state is called *partially synchronized*. As K is increased r_∞ and therefore the synchronized cluster grows. Figure 1 illustrates this.

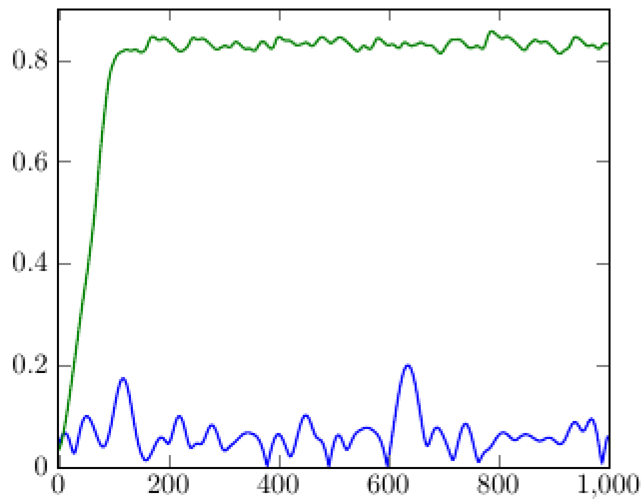

FIGURE 1. A plot of the evolution of $r(t)$ for $K < K_c$ (blue) and $K > K_c$ (green).

Now, such numerical observations should be theoretically justified. In particular, there should be a formula to obtain K_c and $r_\infty(K)$, as well as some stability results for the incoherent and partially synchronized stated. Kuramoto solved the first problem as explained below, but up to now there are only partial results on stability, especially a global stability and convergence result is missing.

For Kuramoto's analysis we have to make some assumptions on the existence of steady state solutions that, of course, have to be justified. First, assume $r(t) = r_\infty$ is constant in time, so that

$$\dot{\theta}_i(t) = \omega_i + K r_\infty \sin\big(\psi(t) - \theta_i(t)\big).$$

Furthermore, let $\psi(t)$ rotate with constant frequency, therefore we can set $\psi(t) = 0$ using a rotating frame of reference. The equations become

$$\dot{\theta}_i(t) = \omega_i - K r_\infty \sin\big(\theta_i(t)\big).$$

In this system, the oscillators are effectivly independent and we can deduce two different types of large time behaviour.

(1) If $|\omega_i| \le K r_\infty$, then the solution θ_i approaches a stable fixed point implicitly defined by $\omega_i = K r_\infty \sin(\theta_i)$.
(2) If $|\omega_i| > K r_\infty$, then θ_i is drifting along the circle with a different velocity.

These are exactly the two groups described above. Kuramoto computed an explicit formula for r_∞ which is in excellent agreement with numerical simulations. More precisely in the limit of large N it holds that

$$r_\infty = \sqrt{1 - \frac{K_c}{K}}, \quad K_c = \frac{2}{\pi g(0)}. \tag{7.2}$$

7.2. A Linear Reformulation of the Kuramoto Model

We mentioned that despite its simple formulation, the Kuramoto model lacks a global stability result. In this section we study a linear reformulation in terms of the variables $Z_t^i = (\cos(\theta_i(t)), \sin(\theta_i(t))) \in \mathbb{R}^2$. Using the notation $\hat{z} = z^\perp = (-y, x)$ for $z = (x, y)$ this is given by

$$\frac{d}{dt} Z_t^i = \omega_i \hat{Z}_t^i + \frac{\theta}{N} \sum_{j=1}^N \left(Z_t^j - Z_t^i\right) - \frac{\theta}{N} \sum_{j=1}^N \langle Z_t^j - Z_t^i, Z_t^i \rangle Z_t^i. \tag{7.3}$$

Here, the drift $b(\omega, z) := \omega \hat{z}$ yields the rotation, the coupling is linear and all-to-all and we have to introduce a third term that forces Z_t^i onto the unit circle, i. e. $\|Z_t^i\| = 1$ for all $t \geq 0$ if $\|Z_0^i\| = 1$. This can be seen as a projection onto the unit circle.

Lemma 7.1. *Let $\|Z_0^i\| = 1$, $i = 1, \dots, N$. Then $\|Z_t^i\| = 1$, $i = 1, \dots, N$ for all $t \geq 0$.*

Proof. Let $i = 1, \dots, N$ and consider

$$\begin{aligned} \frac{d}{dt} \|Z_t^i\|^2 &= 2\omega_i \langle \hat{Z}_t^i, Z_t^i \rangle + \frac{2\theta}{N} \sum_{j=1}^N \langle Z_t^j - Z_t^i, Z_t^i \rangle \left(1 - \|Z_t^i\|^2\right) \\ &=: f_t^i \left(1 - \|Z_t^i\|^2\right). \end{aligned}$$

The differential equation $\dot{g}_t = f_t(1 - g_t)$ has the solution

$$g_t = 1 - (1 - g_0) \mathrm{e}^{\int_0^t f_s \, ds}.$$

Thus, if $g_0 = 1$ we have shown, that $g_t = 1$ for all $t \geq 0$. □

We have verified that (7.3) looks like the canonical linear (except for the projection term) analogue of the Kuramoto model. Also, this formulation looks even simpler and should allow for an explicit stability analysis with the tools developed in Chapter 5. First, let us consider the special case of N identical oscillators.

Proposition 7.2. *Let $\omega_i = \omega$ for all $i = 1, \dots, N$. Then* (7.3) *synchronizes with exponential rate θ.*

Proof. Next observe that

$$\begin{aligned} \frac{d}{dt} \|Z_t^i - Z_t^j\|^2 &= 2 \langle Z_t^i - Z_t^j, \dot{Z}_t^i - \dot{Z}_t^j \rangle \\ &= 2\omega \langle Z_t^i - Z_t^j, \hat{Z}_t^i - \hat{Z}_t^j \rangle + 2 \langle Z_t^i - Z_t^j, \frac{\theta}{N} \sum_{k=1} (Z_t^k - Z_t^i) - (Z_t^k - Z_t^j) \rangle \\ &\quad - \frac{2\theta}{N} \sum_{k=1}^N \langle Z_t^k - Z_t^i, Z_t^i \rangle \langle Z_t^i, Z_t^i - Z_t^j \rangle \\ &\quad + \frac{2\theta}{N} \sum_{k=1}^N \langle Z_t^k - Z_t^j, Z_t^j \rangle \langle Z_t^j, Z_t^i - Z_t^j \rangle. \end{aligned}$$

Note that on the unit circle, the distance depends only on the angle between two vectors since the norms are fixed. This implies for $x, y \in \mathbb{R}^2$, $\|x\| = \|y\| = 1$ that

$$\langle x-y, y\rangle = \langle x, y\rangle - \|y\|^2 = \langle x, y\rangle - 1 = \langle x, y\rangle - \frac{1}{2}\big(\|x\|^2 + \|y\|^2\big) = -\frac{1}{2}\|x-y\|^2.$$

Thus, it follows that

$$\frac{d}{dt}\|Z_t^i - Z_t^j\|^2 = -2\theta\|Z_t^i - Z_t^j\|^2 + \frac{\theta}{2N}\sum_{k=1}^{N}\|Z_t^k - Z_t^i\|^2\|Z_t^i - Z_t^j\|^2 + \|Z_t^k - Z_t^j\|^2\|Z_t^i - Z_t^j\|^2.$$

Now consider the sum over all i and j, hence

$$\frac{d}{dt}\sum_{i,j=1}^{N}\|Z_t^i - Z_t^j\|^2 \le -2\theta\sum_{i,j=1}^{N}\|Z_t^i - Z_t^j\|^2 + \frac{\theta}{N}\left(\sum_{i,j=1}^{N}\|Z_t^i - Z_t^j\|^2\right)^2.$$

which implies that

$$\sum_{i,j=1}^{N}\|Z_t^i - Z_t^j\|^2 \le \frac{\mathrm{e}^{-2\theta t}\sum_{i,j=1}^{N}\|Z_0^i - Z_0^j\|^2}{1 + \frac{1}{2N}\Big(1 - \mathrm{e}^{-2\theta t}\Big)\sum_{i,j=1}^{N}\|Z_0^i - Z_0^j\|^2} \le \mathrm{e}^{-2\theta t}\sum_{i,j=1}^{N}\|Z_0^i - Z_0^j\|^2.$$

Hence, the system completely synchronizes with exponential rate θ. $\square$

However, the reformulation (7.3) is not as good as it seems at first glance. Consider the case of two subpopulations, i. e. $2N$ oscillators, where $\omega_i = \omega_+ := \omega + \frac{\Delta}{2}$ for $i = 1, \dots, N$ and $\omega_i = \omega_- := \omega - \frac{\Delta}{2}$ for $i = N+1, \dots, N$. One would expect, that using the same reasoning as before, it is possible to show synchronization of each subpopulation. But the problem is the nonlinear projection term, where the interactions between the subpopulations do not cancel out. Below, we show this in detail.

As calculated before

$$\begin{aligned}\frac{d}{dt}\|Z_t^j - Z_t^i\|^2 = -2\theta\|Z_t^j - Z_t^i\|^2 &- \frac{\theta}{N}\sum_{k=1}^{2N}\langle Z_t^k - Z_t^j, Z_t^j\rangle\langle Z_t^j, Z_t^j - Z_t^i\rangle \\ &+ \frac{\theta}{N}\sum_{k=1}^{2N}\langle Z_t^k - Z_t^i, Z_t^i\rangle\langle Z_t^i, Z_t^j - Z_t^i\rangle.\end{aligned}$$

Now the sum involves oscillators of the second subpopulation. All parts of the sum that involve $k = 1, \dots, N$ are treated exactly as before.

$$\begin{aligned}\frac{d}{dt}\sum_{i,j=1}^{N}\|Z_t^j - Z_t^i\|^2 \le -2\theta\sum_{i,j=1}^{N}\|Z_t^j - Z_t^i\|^2 &+ \frac{\theta}{2N}\left(\sum_{i,j=1}^{N}\|Z_t^i - Z_t^j\|^2\right)^2 \\ &+ \frac{\theta}{4N}\sum_{i,j=1}^{N}\sum_{k=N+1}^{2N}\big(\|Z_t^k - Z_t^j\|^2 + \|Z_t^k - Z_t^i\|^2\big)\|Z_t^j - Z_t^i\|^2.\end{aligned}$$

The interaction between oscillators of different subpopulations can be estimated simply by

$$\|Z_t^k - Z_t^j\|^2 \le 2\|Z_t^k\|^2 + 2\|Z_t^j\|^2 \le 4.$$

However, this crude estimate implies

$$\frac{d}{dt}\sum_{i,j=1}^{N}\|Z_t^j - Z_t^i\|^2 \leq \frac{\theta}{2N}\left(\sum_{i,j=1}^{N}\|Z_t^i - Z_t^j\|^2\right)^2,$$

whose solution explodes in finite time. Obviously, this approach is not working and has to be refined but at this point it is not clear how to do that. The overall hope is to dominate the mean of each subpopulation by one oscillator that is coupled with the second one. Then, the analysis of this coupling should yield two cases depending on the relation of θ and Δ: In the first case one has full synchronization, whereas in the second one only the two subpopulations synchronize.

7.3. A Linear Reformulation of the Kuramoto Model II

We have seen the limitations of the reformulation (7.3) that does not yield a synchronization result for two subpopulations of oscillators. In this section we put the guesses made in the end on a more solid ground. Consider again $Z_t^i = \mathrm{e}^{\mathrm{i}\theta_i(t)}$, but this time as a complex variable. We study the linearized version

$$\frac{d}{dt}Z_t^i = \big(\mathrm{i}\omega_i - \gamma\big)Z_t^i + \frac{\theta}{N}\sum_{j=1}^{N}Z_t^j. \tag{7.4}$$

Note that there is a linear coupling hidden and the parameter γ is set appropriately, such that we achieve $\lim_{t\to\infty}|Z_t^i| = 1$. Therefore, only in the asymptotic limit each Z_t^i resembles an oscillator on the unit circle, however (7.4) is purely linear. Of course, such a linear model allows for an explicit solution

$$Z_t = \mathrm{e}^{At}Z_0, \quad A := \text{ diag } (i\omega - \gamma) + \frac{\theta}{N}\mathbb{1}_{\mathbb{R}^N\times\mathbb{R}^N},$$

where $\text{diag }(\omega)$ denotes the diagonal matrix with entries from the vector $\omega \in \mathbb{R}^N$ and $\mathbb{1}_{\mathbb{R}^N\times\mathbb{R}^N} := (1)_{1\leq i,j\leq N}$.

The asymptotic behaviour of the system can be reduced to the eigenvalues of the matrix A. Let us consider again the case of $\omega_i = \omega_0$ for all i. Then, it is easy to see that $e_N := (1,\dots,1)$ is an eigenvector to the eigenvalue $\lambda_N = \mathrm{i}\omega_0 - \gamma + \theta$ and there are $N-1$ eigenvectors of the form $e_{N-1} := (1,-1,0,\dots)$ to the eigenvalue $\lambda_{N-1} = \mathrm{i}\omega_0 - \gamma$. As explained above, we require that $\lim_{t\to\infty}|Z_t^i| = 1$, thus all eigenvalues should have non-positive real parts. This is achieved by setting $\gamma = \theta$. In the asymptotics, all contributions of $\lambda_{N-1} = \dots = \lambda_1$ can be neglected and

$$\lim_{t\to\infty}\|Z_t - \langle Z_0, e_N\rangle e_N \mathrm{e}^{\mathrm{i}\omega_0 t}\| = 0\,,$$

thus all oscillators asymptotically synchronize to their common frequency ω_0. Also, the order parameter r_∞ can be calculated as

$$\begin{aligned} r_\infty &= \lim_{t\to\infty}\frac{1}{N}\left|\sum_{i=1}^{N}\mathrm{e}^{\mathrm{i}\theta_i(t)}\right| = \lim_{t\to\infty}\frac{1}{N}\left|\frac{Z_t^i}{|Z_t^i|}\right| \\ &= \lim\frac{1}{N}\left|\sum_{i=1}^{N}\frac{\langle Z_0,e_N\rangle(e_N)_i}{|\langle Z_0,e_N\rangle||(e_N)_i|}\mathrm{e}^{\mathrm{i}\omega_0 t}\right| = 1. \end{aligned}$$

This is compatible to Kuramoto's prediction (7.2), since $g(\omega)\,d\omega = d\delta_{\omega_0}(\omega)$.

Let us get back to the example of two subpopulations. Set the system size to $2N$ and set $\omega_i = \omega_+ := \omega_0 + \frac{\Delta}{2}$ for $1 \le i \le N$ and $\omega_i = \omega_- := \omega_0 - \frac{\Delta}{2}$ for $N+1 \le i \le 2N$. Again, one can check easily that the vector e_{2N} defined by

$$\begin{aligned}
(e_{2N})_i &= \frac{\mathrm{i}\Delta + \theta + \sqrt{\theta^2 - \Delta^2}}{-\mathrm{i}\Delta + \theta + \sqrt{\theta^2 - \Delta^2}}, && 1 \le i \le N, \\
(e_{2N})_i &= 1, && N+1 \le i \le 2N,
\end{aligned}$$

is an eigenvector for A to the eigenvalue $\lambda_{2N} = \mathrm{i}\omega - \gamma + \frac{\theta}{2} + \frac{\sqrt{\theta^2-\Delta^2}}{2}$. Also, e_{2N-1} defined by

$$\begin{aligned}
(e_{2N-1})_i &= -\frac{\mathrm{i}\Delta + \theta - \sqrt{\theta^2 - \Delta^2}}{\mathrm{i}\Delta - \theta + \sqrt{\theta^2 - \Delta^2}}, && 1 \le i \le N, \\
(e_{2N-1})_i &= 1, && N+1 \le i \le 2N,
\end{aligned}$$

is an eigenvector to $\lambda_{2N-1} = \mathrm{i}\omega - \gamma + \frac{\theta}{2} - \frac{\sqrt{\theta^2-\Delta^2}}{2}$. All other eigenvalues are both $(N-1)$-times $\lambda_+ = \mathrm{i}(\omega + \frac{\Delta}{2}) - \gamma$ and $\lambda_- = \mathrm{i}(\omega - \frac{\Delta}{2}) - \gamma$, again with eigenvectors of the form $(1, -1, 0, \dots)$, $(\dots, 0, 1, -1)$.

As guessed before, the asymptotics should depend on the relation between θ and Δ. This is obvious in the linear analysis and the eigenvalues suggest the two cases $\Delta > \theta$ and $\Delta < \theta$. In the first case $\mathrm{Real}\,(\lambda_{2N}) = \mathrm{Real}\,(\lambda_{2N-1}) = -\gamma + \frac{\theta}{2}$, thus $\gamma = \frac{\theta}{2}$ and there are two eigenvalues with real part 0. We obtain

$$\lim_{t\to\infty} \|Z_t - \langle Z_0, e_{2N}\rangle e_{2N} \mathrm{e}^{\mathrm{i}(\omega_0 + \frac{\sqrt{\Delta^2-\theta^2}}{2})t} - \langle Z_0, e_{2N-1}\rangle e_{2N-1} \mathrm{e}^{\mathrm{i}(\omega_0 - \frac{\sqrt{\Delta^2-\theta^2}}{2})t}\| = 0$$

and have no full synchronization among all oscillators, but the two subpopulations synchronize to different collective frequencies.

In the second case, $\sqrt{\theta^2 - \Delta^2}$ is real, thus there is only one eigenvalue with real part 0 by choosing $\gamma = \frac{\theta}{2} + \frac{\sqrt{\theta^2-\Delta^2}}{2}$. Note that $\lim_{\theta\to\infty} \gamma = \theta$ as in the case of identical oscillators. It follows that

$$\lim_{t\to\infty} \|Z_t - \langle Z_0, e_{2N}\rangle e_{2N} \mathrm{e}^{\mathrm{i}\omega_0 t}\| = 0$$

and there is asymptotic synchronization. The order parameter r_∞ is then given by

$$r_\infty = \sqrt{\frac{1 + \sqrt{1 - \frac{\Delta^2}{\theta^2}}}{2}},$$

in agreement with Kuramoto's findings.

7.4. Compartmental Models for Myelinated Neurons

In this section we study an example where the nearest-neighbor coupling between neurons is used. However, this does not correspond to a network of neurons but rather a single neuron with an additional space component, so-called *spatially extended*. In contrast to the models studied before, which are called *space-clamped*, we incorporate the dynamics of the action potential along the axon of a neuron. In principle, the geometry of the axon is assumed to be a uniform, cylindrical electrical wire. Furthermore, all nerve cells can be divided into two major classes depending on the additional properties of their axon: *Myelinated neurons* (majority) and *non-myelinated neurons*.

Myelin is some lipid material that covers and insulates the axon of a myelinated neuron with small, periodic gaps called nodes of Ranvier, where the axon is exposed to the extracellular medium. Non-myelinated neurons do not have such covering. The myelinated parts can be seen as a simple resistor and thus, the propagation of the action potential differs for the two classes of neurons.

- *Saltatory propagation*, i. e. the potential jumps from node to node.
- *Continuous propagation* as in an electrical wire.

In this section we study a model for i. using a compartments that partition the axon of a neuron into different sections and neglect the dendritic tree, the soma and the presynaptic terminals. In each compartment, the membrane potential is modelled with a system of ordinary differential equations describing the space-clamped evolution, such as the Hodgkin-Huxley or FitzHugh-Nagumo equations. Then, nearby compartments are coupled.

7.4.1. Derivation of the Equations. In a few steps one can show that the nearest-neighbor coupling is indeed physically reasonable. Usually, one uses a diagram of equivalent electrical circuits of the axon, see Figure 2.

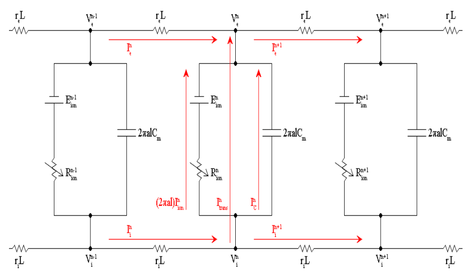

FIGURE 2. The equivalent circuits at nodes n and $n+1$ for the axon of a myelinayed neuron.

First, assume that the nodes of Ranvier are isopotential, i. e. the extra- and intracellular potentials do not vary within the nodes. We call V_e^n and V_i^n the extra- and intracellular potentials in the nth node, respectively.

Next, each myelinated compartment acts as a pure Ohmic resistor, thus

$$V_\mathrm{e}^{n+1} - V_\mathrm{e}^n = -r_\mathrm{e} L I_\mathrm{e}^{n+1},$$
$$V_\mathrm{i}^{n+1} - V_\mathrm{i}^n = -r_\mathrm{i} L I_\mathrm{i}^{n+1},$$

where $r_{\mathrm{e}/\mathrm{i}}$ is the specific extra-/intracellular resistance per unit length and L the length of the myelinated compartment.

Conservation of currents, i. e. Kirchhoff's law, implies

$$I_\mathrm{trans}^n = I_\mathrm{i}^n - I_\mathrm{i}^{n+1} = -I_\mathrm{e}^n + I_\mathrm{e}^{n+1}.$$

Now, define the membrane potential at the nth node as $V^n = V_\mathrm{i}^n - V_\mathrm{e}^n$ and use the relations above to obtain

$$I_\mathrm{trans}^n = \frac{1}{(r_\mathrm{e} + r_\mathrm{i})L}\big(V^{n+1} - 2V^n + V^{n-1}\big).$$

The transmembrane current is given in terms of the usual space-clamped Hodgkin-Huxley or FitzHugh-Nagumo equations, namely

$$I^n_{\mathrm{trans}} = 2\pi a l \Big(C_{\mathrm{m}} \frac{d}{dt} V^n + I_{\mathrm{ion}} \Big),$$

where a is the radius of the axon, l the length of each node of Ranvier and C_{m} the specific membrane capacitance per unit area. Rescaling time by τ_{m}^{-1} we arrive at

$$\frac{d}{dt} V^n_t = \theta\big(V^{n+1}_t - 2V^n_t + V^{n-1}_t \big) - I_{\mathrm{ion}}(t), \tag{7.5}$$

where the parameters are given by

$$\theta = \frac{\lambda_{\mathrm{m}}^2}{lL\tau_{\mathrm{m}}}, \quad \lambda_{\mathrm{m}}^2 = \frac{aR_{\mathrm{m}}}{2\pi a^2 (r_{\mathrm{e}} + r_{\mathrm{i}})}, \quad \tau_{\mathrm{m}} = R_{\mathrm{m}} C_{\mathrm{m}}.$$

τ_{m} and λ_{m} are called the space and time constants of the cable/axon.

So far we have not considered the ends of the axon. One often assumes an infinite axon, i. e. $n \in \mathbb{Z}$, thus the right hand side is always well-defined. If the axon has finite length $n \in \{1, \ldots, N\}$, one has to specify conditions at the boundaries 1 and N. The usual choices are periodic or homogeneous Dirichlet or Neumann boundary conditions, in formula $v^{n+1} = v^1$ and $v^{-1} = v^{n-1}$, $v^0 = v^{n+1} = 0$ or $v^1 - v^0 = v^{n+1} - v^n = 0$.

7.4.2. Synchronization. Equation (7.5) still needs a model for the ionic currents I_{ion}. In the following, we choose the FitzHugh-Nagumo system, i. e. $-I_{\mathrm{ion}} = f(V, W)$ but set the recovery variable W constant equal to the input current I. In this way, there remains only a one-dimensional equation called Nagumo or Bistable equation. Assuming a common additive noise, finite length N and periodic boundary conditions, (7.5) then reads as

$$dV^n_t = \Big(\theta\big(V^{n+1}_t - 2V^n_t + V^{n-1}_t \big) + f\big(V^n_t\big) \Big)\, dt + \sigma\, dW_t, \quad 1 \le n \le N. \tag{7.6}$$

Recall that $f(v) = v(1-v)(v-a)$ and $\sup_{v \in \mathbb{R}} f'(v) \le L^* := \frac{1-a+a^2}{3}$.

PROPOSITION 7.3. *Let $\theta > 2L^* N^2$. Then,* (7.6) *asymptotically synchronizes with exponential rate $\kappa^* := \frac{\theta}{N^2} - 2L^*$.*

For the proof we need the following two lemmas.

LEMMA 7.4 (Summation by Parts). *Let $(a_n), (b_n) \in R^{\mathbb{Z}}$. Then*

$$\sum_{n=1}^N a_n (b_{n+1} - b_n) = a_{N+1} b_{N+1} - a_1 b_1 - \sum_{n=1}^N (a_{n+1} - a_n) b_{n+1}.$$

PROOF. The proof is a simple manipulation.

$$\begin{aligned}
\sum_{n=1}^N a_n (b_{n+1} - b_n) &= \sum_{n=1}^N (a_n - a_{n+1})(b_{n+1} - b_n) + a_{n+1}(b_{n+1} - b_n) \\
&= -\sum_{n=1}^N (a_{n+1} - a_n) b_{n+1} + \sum_{n=1}^N a_{n+1}(b_{n+1} - b_n) + b_n (a_{n+1} - a_n) \\
&= -\sum_{n=1}^N (a_{n+1} - a_n) b_{n+1} + \sum_{n=1}^N a_{n+1} b_{n+1} - b_n a_n.
\end{aligned}$$

The latter is a telescopic sum, thus only the boundary terms remain. □

LEMMA 7.5 (Poincaré Inequality). *Let* $(u_n) \in \mathbb{R}^N$ *such that* $\sum_{n=1}^N u_n = 0$. *Then*

$$\sum_{n=1}^{N-1} (u_{n+1} - u_n)^2 \geq \frac{1}{2N^2} \sum_{n=1}^{N} u_n^2.$$

PROOF.

$$\begin{aligned}
\sum_{n=1}^N u_n^2 &= \sum_{n=1}^N \Big(u_n - \frac{1}{N} \sum_{i=1}^N u_i \Big)^2 = \frac{1}{N^2} \sum_{n=1}^N \Big(\sum_{i=1}^N (u_n - u_i) \Big)^2 \\
&\leq \frac{1}{N} \sum_{n=1}^N \sum_{i=1}^N (u_n - u_i)^2 = \frac{2}{N} \sum_{n=1}^N \sum_{i=1}^{n-1} \Big(\sum_{j=i}^{n-1} (u_{j+1} - u_j) \Big)^2 \\
&\leq \frac{2}{N} \sum_{n=1}^N \sum_{i=1}^{n-1} |n-i| \sum_{j=i}^{n-1} (u_{j+1} - u_j)^2 \\
&= \sum_{j=1}^{N-1} (u_{j+1} - u_j)^2 \Big(\frac{2}{N} \sum_{n=j+1}^{N} \sum_{i=1}^{j} (n-i) \Big) \leq 2N^2 \sum_{j=1}^{N-1} (u_{j+1} - u_j)^2. \qquad \square
\end{aligned}$$

PROOF OF PROPOSITION 7.3. Consider the Lyapunov function $\mathcal{V}(v) := \sum_{n=1}^{N-1} (v^{n+1} - v^n)^2$. Itô's formula yields

$$\begin{aligned}
d\big(\mathrm{e}^{\kappa^* t} \mathcal{V}(V_t)\big) &= \kappa^* \mathrm{e}^{\kappa^* t} \mathcal{V}(V_t)\, dt + 2\mathrm{e}^{\kappa^* t} \sum_{n=1}^{N-1} \big(V_t^{n+1} - V_t^n\big)\big(dV_t^{n+1} - dV_t^n\big) \\
&= \mathrm{e}^{\kappa^* t} \Big(\kappa^* \mathcal{V}(V_t) + 2 \sum_{n=1}^{N-1} \big(f(V_t^{n+1}) - f(V_t^n)\big)\big(V_t^{n+1} - V_t^n\big) \\
&\qquad + 2\theta \sum_{n=1}^{N-1} \big(V_t^{n+1} - V_t^n\big)\big(V_t^{n+2} - 3V_t^{n+1} + 3V_t^n - V_t^{n-1}\big) \Big)\, dt
\end{aligned}$$

because noise and Itô correction cancel out. The drift f is one-sided Lipschitz and coupling term is estimated using the summation by parts and the Poincaré inequality, since for $u_n = V_t^{n+1} - V_t^n$ it holds that $\sum_{n=1}^N u_n = V_t^{N+1} - V_t^1 = 0$ by the periodic boundary. Hence

$$d\big(\mathrm{e}^{\kappa^* t} \mathcal{V}(V_t)\big) \leq \mathrm{e}^{\kappa^* t} \mathcal{V}(V_t) \big(\kappa^* + 2L^* - \theta N^{-2}\big)\, dt \leq 0.$$

This implies $\mathcal{V}(V_t) \leq \mathrm{e}^{-\kappa^* t} \mathcal{V}(V_0) \to 0$ pathwise. $\square$

REMARK 7.6. Note that in the the case $\theta > 2L^*$ the system can be reduced to a one-dimensional system by picking any V_t^n out of the system. More interesting phenomena occur in the opposite case and in the limit $N \to \infty$ with $\tilde{\theta} \leq 2L^*$ using the diffusive scaling $\theta = N^2 \tilde{\theta}$.

Proposition 7.3 states that (7.6) synchronizes, thus $|V_t^n - v_t| \to 0$ as $t \to \infty$ for all $1 \leq n \leq N$ and some fixed process v_t. Obviously,

$$dv_t = f(v_t)\, dt + \sigma\, dW_t,$$

which is the equation for the single, space-clamped neuron. Setting $\sigma = 0$ we can look at the asymptotic behavior of v_t. Since f has three zeros, we also have three equilibria 0, a and 1. Looking at f' we see that 0 and 1 are stable, whereas a is instable. Depending on v_{t_0} for some $t_0 > 0$ we can expect these three cases:

(1) $v_{t_0} < a \Rightarrow \lim_{t\to\infty} v_t = 0$.
(2) $v_{t_0} = a \Rightarrow v_t = a$ for all $t \geq t_0$.
(3) $v_{t_0} > a \Rightarrow \lim_{t\to\infty} v_t = 1$.

As a neurophysiological interpretation, the state $v = 0$ corresponds to a node in the resting state and $v = 1$ to an excited one. Thus, Proposition 7.3 states that the axon is either fully excited or in the resting state, depending on the initial condition.

7.4.3. Traveling Wave Fronts. Synchronization is not a phenomenon that is realistic for the axon of a neuron. Since its purpose is the propagation of a signal to the presynaptic terminal, we study *traveling wave fronts* that connect the two stable states 0 and 1, thus from rest to excited. Due to the simplification to a one-dimensional model—recall that we set the recovery variable W constant—the system is not able to get back to the resting state and form a *pulse*. However, the additional dimension extremely complicates the analysis and we stick to the simpler Nagumo equation that allows for explicit calculations.

In the following, consider $\sigma = 0$ and an infinite axon, i. e. the domain of (7.6) is the lattice $n \in \mathbb{Z}$. Choose an initial condition $V_0 = (V_0^n)_{n\in\mathbb{Z}}$ such that $0 \leq V_0^n \leq 1$ and $\lim_{n\to-\infty} V_0^n = 1$ as well as $\lim_{n\to\infty} V_0^n = 0$. In this case, there appears a traveling wave front defined by the iterative relation

$$V_t^{n+1} = V_{t-\tau_\mathrm{d}}^n, \tag{7.7}$$

where τ_d is the time delay for the wave to jump between successive nodes. The discrete structure makes the proof of existence nontrivial, thus we only cite the following theorem.

THEOREM 7.7. *[See* [**Zin92**]*] There exists $\theta^* > 0$ such that for all $\theta > \theta^*$ the discrete Nagumo equation* (7.6) *admits a traveling wave solution, i. e. $V_t^n = U(n - ct)$, $c > 0$, $U \in C^1(\mathbb{R}, [0,1])$, $\lim_{x\to-\infty} U(x) = 1$, $\lim_{x\to\infty} U(x) = 0$ and $U'(x) < 0$ for all $x \in \mathbb{R}$.*

The function U in the theorem is the *wave profile* that is invariant under (7.6) up to a constant shift with velocity c. From another point of view, this implies $V_t^n = \Phi(t)$ for a certain $\Phi(t) = U(n - ct)$. Now (7.6) and (7.7) together yield

$$\frac{d}{dt}\Phi(t) = \theta\big(\Phi(t-\tau_\mathrm{d}) - 2\Phi(t) + \Phi(t+\tau_\mathrm{d})\big) + f\big(\Phi(t)\big). \tag{7.8}$$

(7.8) is a delay differential equation that is hard to solve. Assuming Φ is sufficently smooth and τ_d sufficiently small we can expand $\Phi(t-\tau_\mathrm{d})$ and $\Phi(t+\tau_\mathrm{d})$ into their Taylor polynomial

$$\Phi(t-\tau_\mathrm{d}) = \Phi(t) - \Phi'(t)\tau_\mathrm{d} + \frac{1}{2}\Phi''(t)\tau_\mathrm{d}^2 + O\big(\tau_\mathrm{d}^3\big),$$
$$\Phi(t+\tau_\mathrm{d}) = \Phi(t) + \Phi'(t)\tau_\mathrm{d} + \frac{1}{2}\Phi''(t)\tau_\mathrm{d}^2 + O\big(\tau_\mathrm{d}^3\big).$$

Thus, the coupling term reduces to $\theta\Phi''(t)\tau_\mathrm{d}^2$ and (7.8) with neglected terms of higher order than τ_d^2 becomes

$$\theta\tau_\mathrm{d}^2\hat{\Phi}''(t) - \hat{\Phi}'(t) + f\big(\hat{\Phi}(t)\big) = 0.$$

We can solve this equation explicitly and for this reason we introduce the new variables $\hat{\Phi}(t) = \hat{U}(-\hat{c}t)$ with $\hat{c}^2 = (\theta\tau_{\mathrm{d}}^2)^{-1}$ that satisfy

$$\hat{U}'' + \hat{c}\hat{U}' + f\big(\hat{U}\big) = 0. \tag{7.9}$$

Note that $\hat{c}$ is not the wave speed from Theorem 7.7, and of course $\hat{U}$ not the wave profile, since (7.9) is only an approximation. To solve this equation we make the following Ansatz:

$$\hat{U}' = -A\hat{U}(1-\hat{U}), \quad A \in \mathbb{R}.$$

Then $\hat{U}'' = A\hat{U}'(2\hat{U}-1)$ and $f(\hat{U}) = -A^{-1}\hat{U}'(\hat{U}-a)$. Therefore

$$\begin{aligned} 0 &= A^2\hat{U}'(2\hat{U}-1) + A\hat{c}\hat{U}' - \hat{U}'(\hat{U}-a) \\ &= \hat{U}'\Big(\big(-A^2 + A\hat{c} + a\big) + \hat{U}\big(2A^2-1\big)\Big), \end{aligned}$$

and setting the coefficients to 0 implies

$$A = \frac{1}{\sqrt{2}} \quad \text{and} \quad \hat{c} = \sqrt{2}\Big(\frac{1}{2} - a\Big). \tag{7.10}$$

It is obvious, that $\hat{c}$ changes its sign as a approaches $\frac{1}{2}$. For $a \in (0, \frac{1}{2})$ we have a wave moving to the right ($c > 0$), $a \in (\frac{1}{2}, 1)$ a wave moving to the left ($c > 0$) and a standing wave ($c = 0$) for $a = \frac{1}{2}$. However, in all cases the wave profile is the same, thus in the setting describing signal propagation in the axon, $c > 0$ is most suitable, since then $\lim_{t\to\infty} V_t^n = 1$ for all n because the wave converts medium from the resting to the excited state.

The Ansatz reduces (7.9) to a first order nonlinear ordinary differential equation that can be solved by separating variables.

$$\int_0^\xi \frac{\hat{U}'(\zeta)}{\hat{U}(\zeta)\big(1-\hat{U}(\zeta)\big)}\,d\zeta = -\frac{\xi}{\sqrt{2}} \quad \Leftrightarrow \quad \int_{\hat{U}(0)}^{\hat{U}(\xi)} \frac{1}{x(1-x)}\,dx = -\frac{\xi}{\sqrt{2}}.$$

By $(\log(x) - \log(1-x))' = \frac{1}{x} + \frac{1}{1-x} = \frac{1}{x(1-x)}$ we arrive at the formula

$$\hat{U}(\xi) = \frac{1}{1 + \exp\big(\frac{\xi}{\sqrt{2}}\big)}. \tag{7.11}$$

Remark 7.8. There are ways to calculate the wave speed more accurately usign higher order corrections, but the approach presented above is simple and has a nice relation to the corresponding equation for non-myelinated neurons. As it turns out, $\hat{c}$ is exactly the wave speed for the traveling wave solution to the continuous analogue. For more detail we refer to the next section.

As already mentioned above, $\hat{c}$ is not the wave speed from Theorem 7.7, however, we can use $\hat{c}$ to approximately determine τ_{d}. Since velocity is length per time unit we know that

$$c = \frac{L+l}{\tau_{\mathrm{d}}} \approx (L+l)\hat{c}\sqrt{\frac{\theta}{\tau_{\mathrm{m}}}} = \frac{L+l}{\sqrt{Ll}}\hat{c}\frac{\lambda_{\mathrm{m}}}{\tau_{\mathrm{m}}}.$$

The space and time constants $\lambda_{\mathrm{m}}, \tau_{\mathrm{m}}$ are more or less independent of the myelination, thus the prefactor $\frac{L+l}{\sqrt{Ll}}$ is responsible for an increased speed. Typical scales are $L \approx 100l$, thus

$$\frac{L+l}{\sqrt{Ll}} \approx \frac{101l}{10l} \approx 10,$$

i. e. in myelinated axons the signal propagates approximately 10 times faster than in non-myelinated ones.

7.5. Lattice Approximation of Stochastic Partial Differential Equations

From a numerical point of view (7.6) looks like a second order approximation of the Laplacian. In this section we make this intuitive idea rigorous.

7.5.1. Finite Difference Approximations. Consider the domain $I = (0, 1)$ and each node x_n is the nth space point in the equidistant grid $I_N := \frac{1}{N}\{0, \dots, N\}$ involving the boundary points. Given a smooth $f : I \to \mathbb{R}$ we want to approximate f' and f'' in terms of $f(x_n)$, $x_n \in I_N$ only.

One-sided differences:

$$\begin{aligned} f(x_{n+1}) &= f(x_n) + f'(x_n)(x_{n+1} - x_n) + O\big(|x_{n+1} - x_n|^2\big) \\ \Rightarrow f'(x_n) &= N\big(f(x_{n+1}) - f(x_n)\big) + O\big(N^{-1}\big), \\ f(x_{n-1}) &= f(x_n) + f'(x_n)(x_{n-1} - x_n) + O\big(|x_{n-1} - x_n|^2\big) \\ \Rightarrow f'(x_n) &= N\big(f(x_n) - f(x_{n-1})\big) + O\big(N^{-1}\big). \end{aligned}$$

It is reasonable to introduce centered differences that have a higher order accuracy. Expand f one more time in its second order Taylor polynomial around x_n, i. e.

$$\begin{aligned} f(x_{n+1}) &= f(x_n) + f'(x_n)(x_{n+1} - x_n) + \frac{1}{2}f''(x_n)(x_{n+1} - x_n)^2 + O\big(|x_{n+1} - x_n|^3\big), \\ f(x_{n-1}) &= f(x_n) + f'(x_n)(x_{n-1} - x_n) + \frac{1}{2}f''(x_n)(x_{n-1} - x_n)^2 + O\big(|x_{n-1} - x_n|^3\big). \end{aligned}$$

Now subtract both equations to obtain

$$f'(x_n) = \frac{N}{2}\big(f(x_{n+1} - f(x_{n-1})\big) + O\big(N^{-2}\big). \tag{7.12}$$

Similarly, on can add both equations to obtain an approximation of the second derivative.

$$f''(x_n) = N^2\big(f(x_{n+1}) - 2f(x_n) + f(x_{n-1})\big) + O\big(N^{-2}\big). \tag{7.13}$$

Recall the similarity with the nearest-neighbor coupling in the diffusive scaling $\theta = \tilde{\theta}N^2$.

Remark 7.9.
(1) The formula (7.13) uses the so-called stencil $(1, -2, 1)$. Higher order approximations are achieved by similar calculations using a stencil with more than three points. The five-point stencil for the second derivative is $(-1, 16, -30, 16, -1)$ with error $O(N^{-4})$.
(2) In the following, we consider I with Neumann boundary conditions representing the current in- and outflow. In order to get a $O(N^{-2})$ approximation in all points, one has to use (7.12) for the normal derivatives in the boundary points 0 and 1.

7.5.2. Spatially Extended FitzHugh-Nagumo Systems. As shown above, the appropriate equations for the axon are (7.5) with coupling $\theta(V^{n+1} - 2V^n + V^{n-1})$. Now, assume the axon is not myelinated, thus it is appropriate to set $l = L = \frac{1}{N}$, i. e. a continuous limit $N \to \infty$ describes the dynamics. Then

$$\theta = \frac{\lambda_{\mathrm{m}}^2}{Ll\tau_{\mathrm{m}}} = \tilde{\theta}N^2 \quad \text{with} \quad \tilde{\theta} = \frac{\lambda_{\mathrm{m}}^2}{\tau_{\mathrm{m}}},$$

i. e. the diffusive scaling but space and time constant of the axon do not change. In the following, we set $\tilde{\theta} = 1$, which can be achieved by a rescaling of the variables.

In order to have an $O(N^{-2})$ approximation we choose a centered difference approximation of the normal derivative in the boundary points 0 and 1. This introduces artificial variables V^{-1} and V^{N+1}, because

$$\frac{N}{2}(V^1 - V^{-1}) = 0, \quad \frac{N}{2}(V^{N+1} - V^{N-1}) = 0.$$

Together with the formula for the second derivative, we can eliminate these variables

$$\begin{aligned} N^2(V^1 - 2V^0 + V^{-1}) &= 2N^2(V^1 - V^0), \\ N^2(V^{N+1} - 2V^N + V^{N-1}) &= -2N^2(V^N - V^{N-1}). \end{aligned}$$

In this section we choose $I^n_{\text{ion}}(t) = f(V^n_t, W^n_t)$, i. e. the FitzHugh-Nagumo nonlinearity $f(v,w) = v(1-v)(v-a) - w$ with the additional dynamics for the recovery variable w as our favorite model for the ionic currents. Writing $V_t = (V^0_t, \ldots, V^N_t) \in \mathbb{R}^{N+1}$, $W_t = (W^0_t, \ldots, W^N_t) \in \mathbb{R}^{N+1}$ and

$$A := N^2 \begin{pmatrix} -2 & 2 & 0 & \ldots & 0 \\ 1 & -2 & 1 & 0 & \ldots \\ & \ddots & \ddots & \ddots & \\ \ldots & 0 & 1 & -2 & 1 \\ 0 & \ldots & 0 & 2 & -2 \end{pmatrix} \in \mathbb{R}^{(N+1)\times(N+1)},$$

we arrive at the system

$$\begin{aligned} dV_t &= \Big(AV_t + f\big(V_t, W_t\big)\Big)\,dt + \sqrt{Q}\,dB_t \\ dW_t &= \epsilon\big(V_t - \gamma W_t\big)\,dt \end{aligned} \tag{7.14}$$

in vector notation. Here, we introduced additive noise in terms of a $(N+1)$-dimensional Brownian motion B with covariance matrix given by a symmetric, positive definite Q. This can be seen as an intermediate scenario between the cases of independent noise and common noise that we both studied before.

Remark 7.10. When using one-sided differences for the Neumann boundary condition, the matrix A becomes symmetric, which has some advantages, e. g. the summation by parts formula. Clearly, A in our case is non-symmetric. However, introduce the equivalent norm

$$|v|^2_N := \frac{1}{2N}\big((v^0)^2 + (v^N)^2\big) + \frac{1}{N}\sum_{n=1}^{N-1}(v^n)^2$$

on $\mathbb{R}^{N+1}$ with associated inner product $\langle\cdot,\cdot\rangle_N$. Then, A is again symmetric and we have

$$\langle Av, u\rangle_N = -N\sum_{n=1}^{N}(v^n - v^{n-1})(u^n - u^{n-1}), \quad \forall u, v \in \mathbb{R}^{N+1}.$$

The corresponding norm to this bilinear form is given by

$$\|v\|^2_N := N\sum_{n=1}^{N}(v^n - v^{n-1})^2.$$

Furthermore, we need the following generalization to exponents $p \geq 1$, with $|v|_{N,2} = |v|_N$:

$$|v|_{N,p}^p := \frac{1}{2N}\big((v^0)^p + (v^N)^p\big) + \frac{1}{N}\sum_{n=1}^{N-1}(v^n)^p.$$

7.5.3. A Priori Estimates. An essential tool in the analysis of such equations are so-called a priori estimates, i. e. certain estimates of the solution that only depend on the initial condition. In the following we derive three lemmas, which are all useful for different purposes and mimic some standard procedures.

LEMMA 7.11. *For every $t \geq 0$ it holds that*

$$\mathbb{E}\left[|V_t|_N^2 + |W_t|_N^2 + 2\int_0^t \|V_s\|_N^2\,ds + 2\alpha\int_0^t |V_s|_{N,4}^4\,ds\right] \leq \mathrm{e}^{2Lt}\Big(\mathbb{E}\left[|V_0|_N^2 + |W_0|_N^2\right] + t\frac{tr\ Q}{N}\Big).$$

PROOF. Apply Itô's formula to $(V_t^k)^2 + (W_t^k)^2$ for $k = 0, \dots, N$

$$\begin{aligned}(V_t^k)^2 + (W_t^k)^2 &= (V_0^k)^2 + (W_0^k)^2 + 2\int_0^t \big(AV_s\big)_k V_s^k + f\big(V_s^k, W_s^k\big)V_s^k\,ds\\ &+ 2\epsilon\int_0^t V_s^k W_s^k - \gamma(W_s^k)^2\,ds + 2\int_0^t V_s^k\sum_{l=0}^N\big(\sqrt{Q}\big)_{kl}\,dB_s^k + t\sum_{l=0}^N\big(\sqrt{Q}\big)_{kl}^2.\end{aligned}$$

Now we do the summation over k to get $|V_t|_N^2$ and we need the following properties:

- Summation by parts $\langle Av, v\rangle_N = -\|v\|_N^2$,
- recurrence for the nonlinear drift, i. e. there exists $L, \alpha > 0$ such that

$$f(v,w)v + \epsilon vw - \epsilon\gamma w^2 \leq L(v^2 + w^2) - \alpha v^4$$

- and the Frobenius matrix norm is given in terms of the trace

$$\sum_{k,l=0}^N\big(\sqrt{Q}\big)_{kl}^2 = \|\sqrt{Q}\|_F^2 = \text{ tr }\big(\sqrt{Q}\sqrt{Q}^{\perp}\big) = \text{ tr } Q.$$

Together with an integration with respect to $\mathbb{P}$ this yields

$$\begin{aligned}&\mathbb{E}\left[|V_t|_N^2 + |W_t|_N^2 + 2\int_0^t \|V_s\|_N^2\,ds + 2\alpha\int_0^t |V_s|_{N,4}^4\,ds\right]\\ &\quad\leq \mathbb{E}\left[|V_0|_N^2 + |W_0|_N^2 + 2L\int_0^t |V_s|_N^2 + |W_s|_N^2\,ds\right] + t\frac{\text{tr } Q}{N},\end{aligned}$$

since the Itô integral has zero mean. Gronwall's lemma implies the result. ☐

We can improve the result concerning the recovery variable W as follows.

LEMMA 7.12. *For every $T > 0$ it holds that*

$$\mathbb{E}\left[\sup_{t\in[0,T]}\|W_t\|_N^2\right] \leq \mathbb{E}\left[\|W_0\|_N^2\right] + \frac{\epsilon}{4\gamma}\mathrm{e}^{2LT}\Big(\mathbb{E}\left[|V_0|_N^2 + |W_0|_N^2\right] + T\frac{tr\ Q}{N}\Big).$$

PROOF. Since the equation for W_t^k is linear, we can calculate directly

$$
\begin{aligned}
|W_t^k - W_t^{k-1}|^2 &= |W_0^k - W_0^{k-1}|^2 - 2\epsilon\gamma \int_0^t \left(W_s^k - W_s^{k-1}\right)^2 ds \\
&\quad + 2\epsilon \int_0^t \left(W_s^k - W_s^{k-1}\right)\left(V_s^k - V_s^{k-1}\right) ds \\
&\leq |W_0^k - W_0^{k-1}|^2 + \frac{\epsilon}{2\gamma} \int_0^t \left(V_s^k - V_s^{k-1}\right)^2 ds.
\end{aligned}
$$

Hence a summation over k and Lemma 7.11 yields the result. □

For the last lemma in this section we exploit a useful strategy to obtain exponential moment estimates.

LEMMA 7.13. *Let $\beta > 0$ and $T > 0$ be arbitrary and $C = \mathrm{e}^{2LT}(1 + \epsilon(4\gamma)^{-1})$. Then, it holds that*

$$
\begin{aligned}
&\mathbb{E}\left[\exp\left(\beta\Big(2\int_0^T \|V_t\|_N^2 + \alpha|V_t|_{N,4}^4 \, dt + \sup_{t\in[0,T]} \|W_t\|_N^2\Big)\right)\right] \\
&\quad \leq \mathrm{e}^{\frac{T\beta C \;tr\; Q}{3N} + \frac{T\beta(C \;tr\; Q)^2}{3\alpha N^2}} \mathbb{E}\left[\exp\left(2\beta\|W_0\|_N^2 + 2\beta C\big(|V_0|_N^2 + |W_0|_N^2\big)\right)\right]^{\frac{2}{3}}.
\end{aligned}
$$

PROOF. Note that Itô's formula in the proof of Lemma 7.11 and Gronwall's lemma imply

$$
2\int_0^T \|V_t\|_N^2 + \alpha|V_t|_{N,4}^4 \, dt \leq \mathrm{e}^{2LT}\Big(|V_0|_N^2 + |W_0|_N^2 + T\frac{\mathrm{tr}\; Q}{N} + M_T\Big),
$$

where M_T denotes the stochastic integral. Also, we have seen in the proof of Lemma 7.12 that

$$
\sup_{t\in[0,T]} \|W_t\|_N^2 \leq \|W_0\|_N^2 + \frac{\epsilon}{2\gamma}\int_0^T \|V_t\|_N^2 \, dt.
$$

Since the exponential function is monotone increasing, a combination of the two inequalities yields for any $\beta > 0$

$$
\begin{aligned}
&\exp\left(\beta\Big(\int_0^T \|V_t\|_N^2 + \alpha|V_t|_{N,4}^4 \, dt + \sup_{t\in[0,T]} \|W_t\|_N^2\Big)\right) \\
&\quad \leq \exp\left(\beta\|W_0\|_N^2 + \beta C\big(|V_0|_N^2 + |W_0|_N^2 + T\frac{\mathrm{tr}\; Q}{N})\right) \exp\left(\beta C M_T\right),
\end{aligned}
$$

with $C = \mathrm{e}^{2LT}(1 + \epsilon(4\gamma)^{-1})$. Now recall that for every continuous local martingale M vanishing at $t = 0$ the processes

$$
Z_t^\lambda := \exp\Big(\lambda M_t - \frac{\lambda^2}{2}\langle M\rangle_t\Big), \quad \lambda > 0,
$$

is again a continuous local martingale. Thus, by Fatou's lemma a supermartingale and therefore $\mathbb{E}\left[Z_t^\lambda\right] \leq \mathbb{E}\left[Z_0^\lambda\right] = 1$ for all $t \geq 0$. With this information we can derive

$$\begin{aligned}\mathbb{E}\left[\exp\Big(\frac{\lambda}{2}M_t\Big)\right] &= \mathbb{E}\left[\exp\Big(\frac{\lambda}{2}M_t - \frac{\lambda^2}{4}\langle M\rangle_t\Big)\exp\Big(\frac{\lambda^2}{4}\langle M\rangle_t\Big)\right] \\ &\le \mathbb{E}\left[Z_t^\lambda\right]^{\frac{1}{2}}\,\mathbb{E}\left[\exp\Big(\frac{\lambda^2}{4}\langle M\rangle_t\Big)\right]^{\frac{1}{2}} \\ &\le \mathbb{E}\left[\exp\Big(\frac{\lambda^2}{4}\langle M\rangle_t\Big)\right]^{\frac{1}{2}}.\end{aligned}$$

The quadratic variation $\langle M\rangle_t$ can be calculated explicitly in terms of the integrals on the left hand side of our inequality using Young's inequality.

$$\begin{aligned}\langle M\rangle_T &= \int_0^T \left(\frac{1}{4N^2}\big((V_t^0)^2 + (V_t^N)^2\big) + \frac{1}{N^2}\sum_{k=1}^{N-1}(V_t^N)^2\right)\left(\sum_{l=0}^{N}\big(\sqrt{Q}\big)_{kl}^2\right)dt \\ &\le \frac{\mathrm{tr}\ Q}{N}\int_0^T |V_t|_N^2\,dt \le \frac{4\beta\alpha}{\lambda^2}\int_0^T |V_t|_{N,4}^4\,dt + \frac{T\lambda^2}{16\alpha\beta}\Big(\frac{\mathrm{tr}\ Q}{N}\Big)^2.\end{aligned}$$

This implies with $\lambda = 4\beta C$ the estimate

$$\begin{aligned}&\mathbb{E}\left[\exp\left(\beta\Big(\int_0^T \|V_t\|_N^2 + \alpha|V_t|_{N,4}^4\,dt + \sup_{t\in[0,T]}\|W_t\|_N^2\Big)\right)\right]^2 \\ &\quad\le \mathbb{E}\left[\exp\left(\beta\|W_0\|_N^2 + \beta C\big(|V_0|_N^2 + |W_0|_N^2 + T\frac{\mathrm{tr}\ Q}{N}\big)\right)\exp\left(\beta C M_T\right)\right]^2 \\ &\quad\le \mathbb{E}\left[\exp\left(2\beta\|W_0\|_N^2 + 2\beta C\big(|V_0|_N^2 + |W_0|_N^2 + T\frac{\mathrm{tr}\ Q}{N}\big)\right)\right]\mathbb{E}\left[\exp\left(2\beta C M_T\right)\right] \\ &\quad\le \mathbb{E}\left[\exp\left(2\beta\|W_0\|_N^2 + 2\beta C\big(|V_0|_N^2 + |W_0|_N^2 + T\frac{\mathrm{tr}\ Q}{N}\big)\right)\right] \\ &\qquad\times\mathbb{E}\left[\exp\left(\beta\alpha\int_0^T |V_t|_{N,4}^4\,dt + \frac{T\beta^2C^2}{\alpha\beta}\Big(\frac{\mathrm{tr}\ Q}{N}\Big)^2\right)\right]^{\frac{1}{2}}.\end{aligned}$$

Now, we can absorb the time integral on the right hand side by the one on the left hand side. This immediately yields the result. □

7.5.4. The Limit $N \to \infty$. Since we proposed (7.14) as a discrete approximation for a continuous axon, it is reasonable to consider what happens in the limit $N \to \infty$. For this reason, we should add a superscript N to all variables and matrices, i. e. $V^N = (V^{N,0}, \ldots, V^{N,N})$, A^N, Q^N .... Lemmas 7.11-7.13 suggest that in the case where all Q^N have bounded eigenvalues, i. e.

$$\sup_{N\ge1}\sup_{0\le k\le N}\lambda_k^N \le C,$$

hence $\sup_{N\ge1} N^{-1}$ tr $Q^N \le C$, the a priori estimates are uniform in the approximation parameter N. This is indeed useful to define the limiting object, which will be a function values stochastic process.

Let us first identify a vector $v \in \mathbb{R}^{N+1}$ with the piecewise linear polygon

$$\tilde{v}(x) := (Nx - k + 1)v^k + (k - Nx)v^{k-1}, \quad x \in \Big[\frac{k-1}{N}, \frac{k}{N}\Big].$$

With this linear interpolation, the vector valued processes V_t^N, W_t^N can be identified as function values stochastic processes. In order to be able to let $N \to \infty$ we need to find a suitable function space for our equation. Clearly, one one suppose "$V_t^\infty \in C^2((0,1)) \cap C^1([0,1])$" since in the limiting equation there appears a second derivative with Neumann boundary conditions. But $\tilde{V}_t^N$ is not even in $C^1((0,1))$, however weakly differentiable. This motivates the use of the following spaces:

- The usual L^p-spaces, $p \geq 1$.

  $$L^p(0,1) := \Big\{ u \in \mathcal{B}(0,1) : \int_0^1 |u(x)|^p \, dx < \infty \Big\} \setminus \Big\{ u = 0 \ \ dx\text{-a. e.} \Big\},$$

  i. e. the quotient space with respect to the kernel of the semi-norm

  $$\|u\|_{L^p} := \left(\int_0^1 |u(x)|^p \, dx \right)^{\frac{1}{p}}.$$

  Although technically L^p consists of equivalence classes of functions, they are usually referred to as L^p-functions. Note that each L^p is a (separable) Banach space and $H := L^2(0,1)$ a Hilbert space with inner product $\langle u, v \rangle_H := \int_0^1 u(x)v(x) \, dx$.
- A reasonable generalization of differentiability in the L^p-context is given by the Sobolev spaces

  $$W^{k,p}(0,1) := \{ u \in L^p(0,1) : \|u\|_{W^{k,p}} < \infty \},$$

  where

  $$\|u\|_{W^{k,p}} := \sum_{i \leq k} \|u^{(i)}\|_{L^p}$$

  and $u^{(i)}$ denotes the ith (weak) derivative. Note that $\tilde{V}_t^N$ is differentiable almost everywhere, thus provided enough integrability $\tilde{V}_t^N \in W^{1,p}(0,1)$.

The discrete norms $|\cdot|_{N,p}$ and $\|\cdot\|_N$ have their counterpars in the continuous case. It holds that for $v \in \mathbb{R}^{N+1}$

$$\begin{aligned}\|\tilde{v}\|_{L^2}^2 &= \int_0^1 |\tilde{v}(x)|^2 \, dx = \sum_{k=1}^N \int_{\frac{k-1}{N}}^{\frac{k}{N}} \left((Nx - k + 1)v^k + (k - Nx)v^{k-1} \right)^2 dx \\ &= \frac{1}{3N} \sum_{k=1}^N \left((v^k)^2 + v^k v^{k-1} + (v^{k-1})^2 \right) \leq |v|_{N,2}^2.\end{aligned}$$

Also

$$\|\tilde{v}'\|_{L^2}^2 = \int_0^1 |\tilde{v}'(x)|^2 \, dx = \sum_{k=1}^N \int_{\frac{k-1}{N}}^{\frac{k}{N}} N^2 (v^k - v^{k-1})^2 \, dx = \|v\|_N^2.$$

Hence our a priori estimates (using a slight generalization) translate into

$$\begin{aligned}\tilde{V}_t^N &\in C([0,T]; H) \cap L^2([0,T]; W^{1,2}) \\ \tilde{W}_t^N &\in C([0,T]; H) \cap C([0,T]; W^{1,2}).\end{aligned}$$

One way to obtain the limit $N \to \infty$ is proving convergence of the distributions, hence tightness, but compactness is the crucial point here. In these function spaces, the closed unit ball is not compact anymore, unlike in the $\mathbb{R}^N$-case. In order to obtain a compact set anyway, one has to use certain compact embeddings in function spaces, e. g. $W^{1,2} \to L^2$, i. e. the closed unit ball with respect to $\|\cdot\|_{W^{1,2}}$ is compact in L^2. At this point we have to refer to a separate lecture on stochastic partial differential equations for the details. Let us however briefly describe the limiting object, namely the equation and solution to

$$\begin{aligned} dV_t &= \big(AV_t + f(V_t, W_t)\big)\, dt + \sqrt{Q}\, dB_t, \\ dW_t &= \epsilon\big(V_t - \gamma W_t\big)\, dt. \end{aligned} \tag{7.15}$$

Here,

- $(A, D(A))$ is the Neumann-Laplace operator, i. e. $A : D(A) \subset H \to H : u \mapsto u''$, $D(A) = W^{2,2}(0, 1)$.
- $Q \in L(H)$ is a bounded, linear operator on H, which is furthermore symmetric, positive definite and $\operatorname{tr} Q < \infty$.
- $B_t := \sum_{k=1}^{\infty} e_k \beta_t^k$ is the so-called cylindrical Wiener process on H, where $(\beta_t^k)_{k\geq 1}$ is a family of iid Brownian motions and $e_k(x) := \sqrt{2}\sin(\pi k x)$ is an orthonormal basis of H.

Remark 7.14. Note that B_t is not an element of H, since

$$\mathbb{E}\left[\|\sum_{k=1}^{N} e_k \beta_t^k\|_H^2\right] = \sum_{k=1}^{N} t\|e_k\|_H^2 \mathbb{E}\left[(\beta_t^k)^2\right] = Nt \xrightarrow{N\to\infty} \infty.$$

However, for any $f \in H$ with $\|f\|_H = 1$ the expression $\langle B_t, f\rangle_H$ is always definable, since

$$\begin{aligned} \mathbb{E}\left[\langle B_t, f\rangle_H \langle B_s, g\rangle_H\right] &= \sum_{k,l=1}^{\infty} \mathbb{E}\left[\beta_t^k \beta_s^l\right] \langle f, e_k\rangle_H \langle g, e_l\rangle_H \\ &= (t \wedge s) \sum_{k=1}^{\infty} \langle f, e_k\rangle_H \langle g, e_k\rangle_H = (t \wedge s)\langle f, g\rangle_H. \end{aligned}$$

In particular, $\langle B_t, f\rangle_H$ is a real valued Brownian motion. Also, the object $\sqrt{Q}B_t \in H$, since the condition $\operatorname{tr} Q < \infty$ yields summability in the now weighted sum. For simplicity, let us assume that Q is diagonal with respect to (e_k) and eigenvalues λ_k^2, then

$$\sqrt{Q}B_t = \sum_{k=1}^{\infty} \lambda_k e_k \beta_t^k,$$

hence

$$\mathbb{E}\left[\|\sqrt{Q}B_t\|_H^2\right] = \sum_{k=1}^{2} \lambda_k^2 \|e_k\|_H^2 \mathbb{E}\left[(\beta_t^k)^2\right] = t\sum_{k=1}^{2} \lambda_k^2 = t \operatorname{tr} Q < \infty.$$

Let us conclude this section with a comment on the relation between the operator Q and the matrices Q^N used in the approximation. The following theorem is essential for this.

Theorem 7.15. *For $Q \in L(H)$ are equivalent:*

(1) *$\operatorname{tr} Q < \infty$,*
(2) *$(\sqrt{Q}v)(x) = \int_0^1 q(x, y)v(y)\, dy$ a. e. for a $q \in L^2((0, 1)^2)$.*

Since q is not continuous, a reasonable discretization is given by the mean of the kernel q over rectangles

$$q_{k,l}^N := \big(|I_l||I_k|\big)^{-1} \int_{I_k} \int_{I_l} q(x,y)\,dy\,dx,$$

with $I_k := (\frac{2k-1}{N}, \frac{2k+1}{N})$ for $1 \le k \le N-1$ and $I_0 := (0, \frac{1}{2N})$, $I_N := (\frac{2N-1}{N}, 1)$. Now recall that $|I_l|^{-\frac{1}{2}} \langle B_t, \mathbb{1}_{I_l} \rangle_H =: \tilde{\beta}_t^l$, $0 \le l \le N$ defines a family of independent real valued Brownian motions. Then

$$\sqrt{Q} B_t \approx \sum_{l=0}^{N} q_{k,l}^N \langle B_t, \mathbb{1}_{I_l} \rangle_H \sum_{l=0}^{N} |I_l|^{-\frac{1}{2}} q_{k,l}^N \tilde{\beta}_t^l = \sqrt{Q^N}\, \tilde{B}_t^N,$$

where $(\sqrt{Q^N})_{kl} := |I_l|^{-\frac{1}{2}} q_{k,l}^N$ is the proper scaling.

7.5.5. The Stochastic Nagumo Equation. Consider equation (7.15) on the whole line $\mathbb{R}$ instead of the finite interval $(0,1)$ and again, set the recovery variable W constant equal to the input current I. In this way we arrive at

$$dV_t = \big(AV_t + f(V_t)\big)\,dt + \sqrt{Q}\,dB_t. \tag{7.16}$$

THEOREM 7.16. *Set $Q = 0$. Then, there exists a traveling wave solution to (7.16), i. e. $V_t(x) = U(x - ct)$ with*

$$U(\xi) = \frac{1}{1 + \exp\left(\frac{\xi}{\sqrt{2}}\right)}$$

and $c = \sqrt{2}(\frac{1}{2} - a)$.

PROOF. The expressions for the wave speed c and the waveprofile U already appeared in the approximate calculation of the wave speed in myelinated neurons. For the continuous analogue everything can be calculated explicitly due to the simple structure of f.

Plug in the Ansatz $V_t(x) = U(x - ct)$ into (7.16) and we get the ordinary differential equation

$$-cU' = U'' + f(U),$$

which is exactly (7.9) and we have already solved this one. $\square$

As a conclusion, the wave speed in myelinated neurons is indeed higher than in non-myelinated ones.

CHAPTER 8

Networks of binary neurons

8.1. The mathematical model

We have seen in the previous Chapter that networks of conductance-based neuronal models become quite complex and its statistical properties are in general too complex to be analysed mathematically rigorously. To get a better hand on the global statistical properties of neural networks one simplifies the neural dynamics in each cell and also the synaptic connections as much as possible. A drastic reduction is the reduction of the possible states of each neurons to the two states 0 (inactive) and 1 (active).

We can then describe the states of a network of N binary neurons at a given time t in terms of a binary vector $n(t) = (n_1(t), \ldots, n_N(t))$ with $n_i(t) \in \{0, 1\}$. The input $u_i(t)$ to the i^{th} neuron in the network is given as

$$u_i(t) = \gamma \sum_{k=1}^{N} J_{ij} n_j(t) - m_i \, , i = 1, \ldots, N \, ,$$

where the connectivity J_{ij} is either 0 or 1, so that the $N \times N$-matrix (J_{ij}) describes the connectivity structure of the network, m_i is some mean input that will be determined later. The connectivity matrix is typically given in terms of realizations of random matrices, e.g. Bernoulli-matrices, i.e., J_{ij} i.i.d. Bernoulli, or random matrices with a given degree structure like, e.g. fixed row sum $\sum_{j=1} J_{ij} \equiv K$. For the moment we will consider any connectivity matrix.

The response of the neuron to the given input $u_i(t)$ will be determined in terms of a nonlinear increasing function $f : \mathbb{R} \to [0, 1]$ as follows: given $n_i(t) = 0$, $f(u_i(t))$ specifies the rate at which the neuron becomes active and conversely, given $n_i(t) = 1$, $1 - f(u_i(t))$ specifies the rate at which the neuron becomes silent. The resulting dynamics of the network is then a time-continuous Markov chain on the state space $I_N = \{0, 1\}^N$ with generator matrix $Q(n, m) = 0$ if $|n - m| \geq 2$ and

$$Q(n, m) = \begin{cases} f(u_i) & \text{if } m - n = e_i \\ 1 - f_i(u_i) & \text{if } m - n = -e_i \, . \end{cases}$$

Here e_i denotes the i^{th} unit vector.

The most prominent example of f is the Heaviside function $f(u) = 1_{\{u \geq \theta\}}$ for some given threshold value. This example has been investigated in extensive numerical simulations in the computational neuroscience literature but on the other hand it lacks regularity and we will therefore also consider as particular examples smooth approximations in terms of sigmoid-functions $f(u) = \frac{1}{1+e^{-\gamma(u-\theta)}}$.

The following little identity turns out to be quite useful:

$$\sum_{m:m-n=\pm e_i} Q(n,m) = (1-n_i)f(u_i) + n_i(1-f(u_i)) = f(u_i) + n_i - 2n_i f(u_i) =: q_i(n)\,.$$

It follows immediately that $Q(n,n) = -\sum_{i=1}^{N} q_i(n) =: q(n)$, to comply with the centralization property of Q-matrices.

The martingale structure

According to the general theory of time-continuous Markov chains, given any observable $G : I_N \to \mathbb{R}$ the process

$$M_t = M_t^G = G\left(n(t)\right) - G\left(n(0)\right) - \int_0^t QG\left(n(s)\right)\, ds\,, t \geq 0$$

is a martingale w.r.t. the natural filtration generated by $n(t)$ with

$$\begin{aligned} E\left(M_t^2\right) = \int_0^t E\Big(&\sum_{i:n_i=0}^{N} f(u_i)\left(G\left(n(s)+e_i\right) - G\left(n(s)\right)\right)^2 \\ &+ \sum_{i:n_i=1}^{N} (1-f(u_i))\left(G\left(n(s)-e_i\right) - G\left(n(s)\right)\right)^2 \Big)\, ds \end{aligned}$$

(see Theorem 8.2.2 and Corollary 2.11 in Chapter 2).
For the particular choice of $G(n) = \pi_i(n) = n_i$, we obtain that

$$M_t^i = n_i(t) - \int_0^t f(u_i(s)) - n_i(s)\, ds\,,$$

since

$$\begin{aligned} Q\pi_i(n) &= \sum_m Q(n,m)m_i \\ &= -q(n)n_i + \left(\sum_{j\neq i} q_j(n)\right) n_i + f(u_i)(1-n_i) \\ &= -q_i(n)n_i + f(u_i)(1-n_i) \\ &= -n_i + f(u_i)\,. \end{aligned}$$

Similarly for $G(n) = \pi_{ij}(n) = n_i n_j$, $i \neq j$, we obtain that

$$M_t^{ij} = n_i(t)n_j(t) - \int_0^t \left(-n_i(s)n_j(s) + f(u_i(s))n_j(s) + f(u_j(s))n_i(s)\right)\, ds\,,$$

since

$$\begin{aligned} Q\pi_{ij}(n) &= -q(n)n_i n_j + \sum_{k\neq i,j} q_k(n)n_i n_j + q_i(n)(1-n_i)n_j + q_j(n)(1-n_j)n_i \\ &= \left(n_i + f(u_i) - 2n_i f(u_i)\right)\left(n_j - 2n_i n_j\right) \\ &\quad + \left(n_j + f(u_j) - 2n_j f(u_j)\right)\left(n_i - 2n_i n_j\right) \\ &= 2n_i n_j - 4n_i n_j + f(u_i)(n_j - 2n_i n_j) - 2f(u_i)n_i n_j + 4n_i n_j f(u_i) \\ &\quad + f(u_j)(n_i - 2n_i n_j) - 2f(u_j)n_i n_j + 4n_i n_j f(u_j) \\ &= -2n_i n_j + f(u_i)n_j + f(u_j)n_i\,. \end{aligned}$$

Using the above information on the martingale structure we can formulate the following dynamical equations for the first and second moments:

$$E(n_i(t)) = \int_0^t E\left(f(u_i(s))) - n_i(s)\right) ds$$

hence

$$\frac{d}{dt}E(n_i(t)) = E(f(u_i(t))) - E(n_i(t)) .$$

And again in a similar manner, for $i \neq j$,

$$E(n_i(t)n_j(t)) = \int_0^t E\left(f(u_i(s))n_j(s) + u_i(n(s))n_i(s) - n_i(s)n_j(s))\right) ds$$

hence

$$\frac{d}{dt}E(n_i(t)n_j(t)) = E(f(u_i(t))n_j(t)) + E(f(u_j(t))n_i(t) - 2E(n_i(t)n_j(t)) .$$

As to be expected, the equations cannot be closed in $n_i n_j$. This is, why now the nonlinearity $f(u_i)$ is expanded w.r.t. the mean (and its variance), to get a closed form expression.

8.1.1. Stationary distributions. As the state space I_N is finite, it follows that the Markov chain n has at least one invariant measure. It turns out that the invariant measure never is unique, in fact we have the following elementary observation on forward invariant subsets of the state space:

LEMMA 8.1. *Any state n contained in*

- $C_- := \{n \mid f(u_i) = 0 \, \forall i\}$ *- the set of zero activity,*
- $C_+ := \{n \mid f(u_i) = 1 \, \forall i\}$ *- the set of maximal activity,*

is absorbing, i.e. $Q(n,m) = 0$ for all $n \in C_\pm$, $n \neq m$. In particular, any probability measure μ supported on $C_\pm$ is an invariant measure for $n(t)$.

PROOF. It is easy to see that $f(u_i) = 0$ (resp. $= 1$) for all i and $n \in C_-$ (resp. $n \in C_+$). This implies the assertion. □

This leaves us with the question whether there are nontrivial invariant sets (resp. nontrivial invariant measures), and if so, how to characterize them. In particular, are there invariant measures related to interesting dynamical states, like the asynchronous irregular state observed in simulations and investigated in the computational neuroscience literature (see [**VS98**]).

The following statements hold true for any invariant measure μ:

- $\int n_i \, d\mu(n) = \int F_i(n) \, d\mu(n)$
- $\int n_i n_j \, d\mu(n) = \frac{1}{2} \int F_i(n) n_j + F_j(n) n_i \, d\mu(n)$, $i \neq j$,

which is an immediate consequence of $\int Qf \, d\mu = 0$.

8.2. Elements of a mean-field theory

Within this section we will investigate the asymptotic statistical properties of binary neural networks for large n under appropriate assumptions on the connectivity matrix $J_{ij} = J_{ij}^N$, $1 \le i, j \le N$. It is observed in numerical simulations that the mean activity

$$\bar{n}(t) := \frac{1}{N}\sum_{i=1}^N n_i(t)$$

converges to some deterministic limit under rather general assumptions.

8.2.1. The law of large numbers.

Scenario 1: Fixed row sum: $J_{ij} = J_{ij}^{(N)}$ such that $\sum_{j=1}^N J_{ij}^{(N)} \equiv K_N$ with $K_N \uparrow \infty$, $m^{(N)} \equiv m$, $f^{(N)}(u) = f(u)$, with $\gamma_N \ge 0$ and $f \in C_b^1$.

In this case we will prove that the dynamics of any ensemble average

$$n^{J^{(N)}} := \frac{1}{|J^{(N)}|}\sum_{j\in J^{(N)}} n_j^{(N)}$$

with $|J^{(N)}| \to \infty$ sufficiently fast is asymptotically equivalent to the solution $m^{(N)}(t)$ of the ordinary differential equation

$$\dot{m}^{(N)}(t) = -m^{(N)}(t) + f(\gamma_N K_N (m(t) - m))\,, m(0) = m_0 \tag{8.1}$$

for suitable initial conditions $n_i(0)$, e.g. $n_i(0)$ i.i.d. with $E(n_i(0)) = m_0$.

In order to formulate the Theorem precisely, let us introduce the following distance:

$$d_N(t) := \sup_{\substack{J\subset\{1,\dots,N\}\\ |J|\ge K_N}} E\left(|n^J(t) - m^{(N)}(t)|^2\right)^{\frac{1}{2}}$$

where $m^{(N)}(t)$ is the solution to the mean field equation (8.1).

THEOREM 8.2. *Under the above assumptions we have that*

$$d_N(t) \le d_N(0) + \sqrt{\frac{t}{K_N}} + (\gamma_N K_N \|f\|_{Lip} + 1)\int_0^t d_N(s)\,ds\, t \ge 0\,.$$

Gronwall's inequality implies in particular,

$$d_N(t) \le \left(d_N(0) + \sqrt{\frac{t}{K_N}}\right) e^{(\gamma_N K_N \|f\|_{Lip}+1)t}\,, t \ge 0\,.$$

Suppose now that $K_N \to \infty$, $\sup_{N\ge 1}\gamma_N K_N < \infty$ and initial conditions $n(0)$ are chosen such that $\lim_{N\to\infty} d_N(0) \to 0$, e.g. $n_i(0)$ i.i.d. with $E(n_i(0)) = n_0$, then for every ensemble average $n^{J^{(N)}}$ with $|J^{(N)}| \ge K_N$ it follows that

$$\lim_{N\to\infty} E\left(|n^{J^{(N)}}(t) - m^{(N)}(t)|^2\right) = 0$$

that is, the time evolution of $n^{J^{(N)}}$ is asymptotically equivalent to the time-evolution of $m^{(N)}(t)$ driven by the mean field equation (8.1).

COROLLARY 8.3. *Under the above assumptions. If $\gamma_N K_N \to \gamma_*$, then*

$$\lim_{N\to\infty} E\left(|n^{J^{(N)}}(t) - m(t)|^2\right) = 0$$

where m is a solution to the ordinary differential equation

$$\dot m(t) = -m(t) + f(\gamma_* m(t) - m)\,, m(0) = m_0\,.$$

PROOF. (of Theorem 8.2) We first consider N fixed. To simplify notations we can drop the dependence on N. Fix a subset $J \subset \{1,\dots,N\}$ with $|J| \geq K_N$. $n^J(t)$ admits the following semimartingale decomposition

$$n^J(t) = n^J(0) + \int_0^t Qn^J(s)\,ds + M_t$$

with

$$\begin{aligned} E\left(M_t^2\right) &= \int_0^t \sum_{i\in J:n_i=0} E\left(f(u_i(s))\left(\frac{1}{|J|}\right)^2\right) ds \\ &\quad + \int_0^t \sum_{i\in J:n_i=1} E\left((1-f(u_i(s)))\left(\frac{1}{|J|}\right)^2\right) ds \\ &= \frac{1}{|J|^2}\int_0^t E\left(\sum_{i\in J}(1-n_i(s))f(u_i(s)) + n_i(s)(1-f(u_i(s)))\right) ds \leq \frac{t}{|J|}\,. \end{aligned}$$

It follows that

$$\begin{aligned} \left(E\left(n^J(t)-m(t)\right)^2\right)^{\frac12} &\leq \left(E\left(n(0)-m(0)\right)^2\right)^{\frac12} \\ &\quad + \left(E\left(\int_0^t Qn^J(s) - (f(\gamma_N K_N m(s) - m) - m(s))\,ds\right)^2\right)^{\frac12} + \left(E\left(M_t^2\right)\right)^{\frac12} \\ &= I + II + III\,. \end{aligned}$$

Let us estimate the three terms separately. From the definition $E(I) \leq d_N(0)$, from the above computations $E(III) \leq \sqrt{\frac{t}{|J|}} \leq \sqrt{\frac{t}{K_N}}$. It remains to estimate the second term:

$$II \leq \int_0^t \left(E\left(Qn^J(s) - f(\gamma_N K_N m(s) - m)\right)^2\right)^{\frac12} ds\,.$$

Clearly,

$$\begin{aligned} Qn^J(s) - (f(\gamma_N K_N m(s) - m) - m(s)) &= \sum_{i\in J:n_i=0} f(u_i(s))\frac{1}{|J|} \\ &\quad - \sum_{i\in J:n_i=1}(1-f(u_i(s)))\frac{1}{|J|}) - (f(K_N(m(s)-\theta)) - m(s)) \\ &= \frac{1}{|J|}\sum_{i\in J} f(u_i(s)) - n^J(s) - (f(\gamma_N K_N(m(s)-\theta)) - m(s))g' \\ &= \frac{1}{|J|}\sum_{i\in J}(f(\gamma_N K_N n_i(s) - m) - f(\gamma_N K_N m(s) - m) - (n^J(s) - m(s)) \\ &= II_a(s) + II_b(s)\,, \end{aligned}$$

so that

$$II \leq \int_0^t \left(E(II_a^2(s))\right)^{\frac{1}{2}} + \left(E(II_b^2(s))\right)^{\frac{1}{2}} \, ds$$
$$\leq (\gamma_N K_N \|f\|_{Lip} + 1) \int_0^t d_N(s) \, ds \, .$$

Here $J_i := \{j \in \{1, \ldots, N\} \mid J_{ij} = 1\}$ denotes the set of presynaptic neurons to neuron i. Moreover we have used that $x \mapsto f(\gamma_N K_N x - m)$ is Lipschitz with Lipschitz constant $\gamma_N K_N \|f\|_{Lip}$, so that

$$\left(E\left(II_a(s)^2\right)\right)^{\frac{1}{2}} \leq \gamma_N K_N \|f\|_{Lip} d_N(s) \, .$$

Combining all three estimates we arrive at

$$d_N(t) \leq d_N(0) + \sqrt{\frac{t}{K_N}} + (\gamma_N K_N \|f\|_{Lip} + 1) \int_0^t d_N(s) \, ds$$

□

8.2.2. The central limit theory. Scenario 1 In the next step we want to study the asymptotic fluctuations in the law of large numbers for ensemble averages. For simplicity we assume that $\gamma_N K_N \equiv \gamma_*$ and denote with $m(t)$ the solution of the ordinary differential equation

$$\dot{m}(t) = -m(t) + f(\gamma_* m(t) - m) \, . \tag{8.2}$$

We then define the standardized ensemble averages

$$n^{j,*}(t) := \sqrt{|J|} \left(n^J(t) - m(t)\right) = \sqrt{|J|} \left(\frac{1}{|J|} \sum_{i \in J} n_i(t) - m(t)\right) .$$

In addition we will assume from now on that $f \in C_b^2$.

THEOREM 8.4. *Let $J^{(N)} \subset \{1, \ldots, N\}$, K_N and $d_N(0)$ be such that $|J^{(N)}| \geq K_N$ and $\lim_{N\to\infty} \frac{N}{|J^{(N)}|} \left(d_N(0) + \frac{1}{\sqrt{K_N}}\right) \to 0$. Suppose that $P \circ \left(n^{|J|,*}(0)\right)^{-1} \to N(m_0, \sigma_0^2)$ weakly (e.g. $n_i(0)$ iid Bernoulli (m_0), hence $\sigma_0^2 = m_0(1 - m_0)$).*

We also assume that $n^{J_{i\cdot}^{(N)}}(t)$, $i = 1, \ldots, N$, become asymptotically uncorrelated in the sense that

$$\lim_{N\to\infty} E\left(\sum_{i \in J^{(N)}} \left(n^{J_{i\cdot}^{(N)}}(t) - m(t)\right)^2\right) = \lim_{N\to\infty} \sum_{i \in J^{(N)}} E\left(\left(n^{J_{i\cdot}^{(N)}}(t) - m(t)\right)^2\right) .$$

Then $n^{J^{(N)},*}(t) \to n_\infty(t)$ weakly on the Skorohod space $\mathcal{D}([0, \infty))$. Here n_∞ is the unique solution of the stochastic differential equation

$$dn_\infty(t) = f(\gamma_* m(t) - m) - n_\infty(t) \, dt + \sigma(t) \, dW(t)$$

where $W(t)$ is a 1-dimensional Brownian motion,

$$\sigma(t) := \sqrt{(1 - m(t))f(\gamma_* m(t) - m) + m(t)(1 - f(\gamma_* m(t) - m)} \, , t \geq 0,$$

and $m(t)$ is the solution of the ordinary differential equation 8.2 with initial condition m_0.

Conjecture: The asymptotic uncorrelatedness is true in the case where $K_N \ll N$, e.g. $K_N = \mathcal{O}(\log N)$.

PROOF. We start again with the semimartingale decomposition

$$n^{J^{(N)},*}(t) = n^{J^{(N)},*}(0) + \int_0^t \sqrt{|J^{(N)}}\left(Q^{(N)}n^{J^{(N)}}(s) - \dot{m}(s)\right)\,ds + M_t^{(N)}$$

where

$$M_t^{(N)} := \sqrt{|J^{(N)}|}\left(n^{J^{(N)}}(t) - n^{J^{(N)}}(0) - \int_0^t Q^{(N)}n^{J^{(N)}}(s)\,ds\right)$$

is a martingale w.r.t. the natural filtration generated by $n(t)$. Our aim is to apply the martingale central limit theorem 2.15. To this end note that $M_t^{(N)} - M_{t-}^{(N)} \neq 0$ if and only if there is a switch in one of the neurons at time t from active to inactive or conversely, so that

$$\sup_{t\le T}\left|M_t^{(N)} - M_{t-}^{(N)}\right| \le \frac{1}{\sqrt{J^{(N)}}} \to 0\,, N\to\infty\,.$$

To derive the limiting behaviour of the variance note that

$$E\left(M_t^{(N),2}\right) = \frac{1}{|J^{(N)}|}\int_0^t E\left(\sum_{i\in J^{(N)}}\left((1-n_i(s))f(u_i^{(N)}(s)) + n_i(s)(1-f(u_i^{(N)}(s)))\right)\,ds\right)\,.$$

Since $|J^{(N)}| \ge K_N$ the previous theorem implies that

$$u_i^{(N)}(s) = \gamma_N \sum_{j:J_{ij}^{(N)}=1} n_j(s) - m \to \gamma_* m(s) - m\,, N\to\infty\,,$$

in $L^1(P)$, hence

$$E\left(M_t^{(N),2}\right) \to \int_0^t \sigma(s)^2\,ds$$

as $N\to\infty$ in $L^1(P)$. Moreover, $M_t^{(N),2} - A_t^{(N)}$, $t\ge 0$, is a martingale, where

$$A_t^{(N)} = \int_0^t \sum_{i\in J^{(N)}} (1-n_i(s))f(u^{(N)}(s)) + n_i(s)(1-f(u_i^{(N)}(s)))\,ds$$

(see Theorem). Theorem 2.15 now implies that

$$\lim_{N\to\infty} M_t^{(N)} = \int_0^t \sqrt{(1-m(s))f(\gamma_* m(s)-m) + m(s)(1-f(\gamma_* m(s)-m)}\,dW(s)$$

weakly on the Skorokhod space $\mathcal{D}([0,\infty))$.

We now turn to the proof of the weak convergence of the drift term. To simplify our analysis we introduce the following notation $X^{(N)} \asymp Y^{(N)}$ for two sequences of real-valued stochastic processes $X_t^{(N)}$ and $Y_t^{(N)}$ if $|X_t^{(N)} - Y_t^{(N)}| \to 0$ in $L^1(P)$.

First note that

$$\sqrt{|J^{(N)}|}\left(Q^{(N)}n^{J^{(N)}}(t) - \dot{m}(t)\right) \asymp \sqrt{|J^{(N)}|}A^{(N)}(t)$$

where

$$A^{(N)}(t) := f(\gamma_* m(t)-m) + \gamma_* f'(\gamma_* m(t)-m)\frac{1}{\sqrt{|J^{(N)}|K_N}}\sum_{i\in J^{(N)}} n^{J_{i\cdot}^{(N)},*}(t) - n^{J^{(N)},*}(t)\,.$$

Indeed, a Taylor expansion of $f(u_i^{(N)}(t))$ in $\gamma_* m(t) - m$ yields

$$\sqrt{|J^{(N)}|}\left(Q^{(N)} n^{J^{(N)}}(t) - A^{(N)}(t)\right) = \frac{1}{\sqrt{|J^{(N)}|}} \sum_{i \in J^{(N)}} f(u_i^{(N)}(t)) - n_i(t)$$
$$- f(\gamma_* m(t) - m) - \gamma_* f'(\gamma_* m(t) - m) \frac{1}{\sqrt{|J^{(N)}|} K_N} \sum_{i \in J^{(N)}} n^{J_{i\cdot}^{(N)},*}(t) + n^{J^{(N)},*}(t)$$
$$= \frac{\gamma_*^2}{2\sqrt{|J^{(N)}|}} \sum_{i \in J^{(N)}} f''(\xi_i^{(N)}(t)) \left(n^{J_{i\cdot}^{(N)}}(t) - m(t)\right)^2$$

for certain values $\xi_i^{(N)}(t)$ between $u_i^{(N)}(t)$ and $\gamma_* m(t) - m$. Using the boundedness of f'' and Theorem 8.4 we conclude that

$$\frac{1}{2\sqrt{|J^{(N)}|}} E\left(\sum_{i \in J^{(N)}} |f''(\xi_i^{(N)}(t))| \left(n^{J_{i\cdot}^{(N)}}(t) - m(t)\right)^2\right) \le C_T \frac{\sqrt{|J^{(N)}|}}{2K_N}\,, t \le T\,,$$

for some constant C_T, depending on f only. Because $\sqrt{J^{(N)}}/K_N \to 0$, as $N \to \infty$, we arrive at the required asymptotic equivalence of the two processes.

In the next step we will show that $A^{(N)} \asymp B^{(N)}$, where

$$B^{(N)}(t) = f(\gamma_* m(t) - m) - n^{J^{(N)},*}(t)\,.$$

Indeed, using the asymptotic uncorrelatedness of $n^{J_{i\cdot}^{(N)}}(t)$, $i = 1, \ldots, N$, we conclude that

$$\lim_{N\to\infty} E\left(\left(A^{(N)}(t) - B^{(N)}(t)\right)^2\right)$$
$$= \gamma_*^2 f'(\gamma_* m(t) - m)^2 \lim_{N\to\infty} \frac{|J^{(N)}|}{K_N} E\left(\sum_{i \in J^{(N)}} \left(n^{J_{i\cdot}^{(N)}}(t) - m(t)\right)^2\right)$$
$$\le \limsup_{N\to\infty} C_T \frac{|J^{(N)}|}{K_N^2}\,, t \le T\,,$$

where C_T again is a constant depending only on f.

Hence $n^{J^{(N)},*} \asymp n^{J^{(N)},*}(0) + \int_0^t B^{(N)}(s)\,ds + M_t^{(N)}$. Next, consider the process

$$\bar n^{J^{(N)},*} := e^{-t} n^{J^{(N)},*}(0) + \int_0^t e^{-(t-s)} f(\gamma_* m(s) - m)\,ds + \int_0^t e^{-(t-s)} dM_s^{(N)}$$
$$:= e^{-t} n^{J^{(N)},*}(0) + \int_0^t e^{-(t-s)} f(\gamma_* m(s) - m)\,ds + M_t^{(N)} - \int_0^t e^{-(t-s)} M_s^{(N)}\,ds\,.$$

Then $\bar n^{J^{(N)},*} = \Phi(M^{(N)})(t)$, where $\Phi : \mathcal{D}([0,\infty)) \to \mathcal{D}([0,\infty))$ is the mapping

$$\Phi(\omega)(t) = e^{-t} n^{J^{(N)},*}(0) + \omega(t) - \int_0^t e^{-(t-s)} \omega(s)\,ds$$

(cf. the proof of Theorem 3.1). Since Φ is continuous w.r.t. the Skorohod metric we conclude that

$$\lim_{N\to\infty} \bar n^{J^{(N)},*} = e^{-t} n^{J^{(N)},*}(0) + \int_0^t e^{-(t-s)} f(\gamma_* m(s) - m)\,ds + \int_0^t e^{-(t-s)} \sigma(s)\,dW(s)$$

weakly, hence

$$\lim_{N\to\infty} n^{J^{(N)},*} = e^{-t} n^{J^{(N)},*}(0) + \int_0^t e^{-(t-s)} f(\gamma_* m(s) - m)\, ds \int_0^t e^{-(t-s)} \sigma(s)\, dW(s)$$

weakly too on the Skorohod space $\mathcal{D}([0,\infty))$. □

8.2.3. Extensions to the case $K_N \equiv K$, $\gamma_N \equiv \gamma_*/K$. In the following remarks we will comment on extensions of previously developed elements of the mean-field theory in Theorem 8.2 and Theorem 8.4 to the case of constant row sum $K_N \equiv K$ for all N. In this case, the assumptions for the LLN obtained in Theorem 8.2 are not satisfied, but numerical simulations still show convergence of the mean activity $\bar{n}^{(N)}(t) = \frac{1}{N}\sum_{i=1}^N n_i(t)$ towards some deterministic limit $m(t)$. Surprisingly it turns out that the limiting equation in this case is no longer given by the mean-field equations derived in the previous subsection. In fact, the driving equation for $m(t)$ becomes more involved and we will derive it the following with a formal derivation.

Our starting point will be again the semimartingale decomposition, this time applied to the mean activity $\bar{n}^{(N)}(t) = \frac{1}{N}\sum_{i=1}^N n_i(t)$:

$$\bar{n}^{(N)}(t) = \bar{n}^{(N)}(0) + \int_0^t \frac{1}{N}\sum_{i=1}^N f(u_i^{(N)}(s)) - n_i(s)\, ds + M_t^{(N)}$$

with a martingale $M_t^{(N)}$ satisfying $E\left(M_t^{(N),2}\right) \le \frac{t}{N}$. Concerning the drift part $\int_0^t \frac{1}{N}\sum_{i=1}^N f(u_i^{(N)}(s)) - n_i(s)\, ds$ we now perform a Taylor expansion of the integrand $\frac{1}{N}\sum_{i=1}^N f(u_i^{(N)}(s)) - n_i(s)$ at the assumed asymptotic limit mean activity $m(s)$. More precisely, let $\mu_1(s) := \gamma_* m(s) - m$. Then

$$\frac{1}{N}\sum_{i=1}^N f(u_i^{(N)}(s)) = \sum_{k=0}^\infty \frac{f^{(k)}(\mu_1(s))}{k!} \frac{1}{N}\sum_{i=1}^N \left(u_i^{(N)}(s) - \mu_1(s)\right)^k .$$

thereby assuming that f is analytic.

Our hope is that for increasing k the terms in the Taylor expansion of f will decrease, so that a partial sum approximation will approximate the drift term in the semimartingale decomposition for the mean activity with increasing accuracy. Let us look at the first terms corresponding to $k = 1, 2$ more closely:

k=1: Assuming $\mu_1(t) \asymp \frac{\gamma_*}{N}\sum_{i=1}^N n_i(t) - m$ we have that

$$\frac{1}{N}\sum_{i=1}^N u_i^{(N)}(t) - \mu_1(t) = \gamma_* \left(\frac{1}{N}\sum_{i=1}^N \frac{1}{K}\sum_{j=1}^N J_{ij} n_j(t) - m(t)\right) .$$

Interchanging the summation and assuming the convergence $m(t) = \lim_{N\to\infty} \frac{1}{N}\sum_{i=1}^N n_i(t)$, we can write

$$\lim_{N\to\infty} \frac{1}{N}\sum_{i=1}^N u_i^{(N)}(t) - \mu_1(t) = \frac{\gamma_*}{K} \lim_{N\to\infty} \frac{1}{N}\sum_{j=1}^N \left(\sum_{i=1}^N J_{ij} - K\right) n_j(t) .$$

We will show that this limit is zero. In fact, we will show in Lemma **??** below that the conditional distribution of $(n_j(t))$, given a typical realization of (J_{ij}), is exchangeable. This implies in particular, using the notation $\bar{J}_{ij} := J_{ij} - \frac{K}{N}$, that

$$E\left(\left(\frac{1}{N}\sum_{j=1}^{N}\sum_{i=1}^{N}\bar{J}_{ij}n_j(t)\right)^2\right) = \frac{1}{N^2}\sum_{j_1,j_2=1}^{N} E\left(\sum_{i=1}^{N}\bar{J}_{ij_1}\sum_{i=1}^{N}\bar{J}_{ij_2}E\left(n_{j_1}(t)n_{j_2}(t)\mid (J_{ij})\right)\right)$$

$$= \frac{1}{N^2}\sum_{j_1=j_2=1}^{N} E\left(\left(\sum_{i=1}^{N}\bar{J}_{ij_1}\right)^2 E\left(n_1(t)\mid (J_{ij})\right)\right)$$

$$+ \frac{1}{N^2}\sum_{j_1\neq j_2=1}^{N} E\left(\sum_{i=1}^{N}\bar{J}_{ij_1}\sum_{i=1}^{N}\bar{J}_{ij_2}E\left(n_1(t)n_2(t)\mid (J_{ij})\right)\right)$$

$$= \frac{1}{N^2}\sum_{j=1}^{N} E\left(\left(\sum_{i=1}^{N}\bar{J}_{ij}\right)^2 E\left(n_1(t)(1-n_2(t))\mid (J_{ij})\right)\right)$$

$$+ E\left(\left(\frac{1}{N}\sum_{j=1}^{N}\sum_{i=1}^{N}\bar{J}_{ij}\right)^2 E\left(n_1(t)n_2(t)\mid (J_{ij})\right)\right)$$

$$= \frac{1}{N^2}\sum_{j=1}^{N} E\left(\left(\sum_{i=1}^{N}\bar{J}_{ij}\right)^2 E\left(n_1(t)(1-n_2(t))\mid (J_{ij})\right)\right),$$

since $\frac{1}{N}\sum_{j=1}^{N}\sum_{i=1}^{N}\bar{J}_{ij} = 0$. Now the right hand side converges to 0 as $N \to \infty$, which implies that

$$\lim_{N\to\infty}\sum_{i=1}^{N} u_i^{(N)}(t) - \mu_1(t) = 0 \text{ in } L^2(P).$$

k=2: Assuming $\mu_1(t) \asymp \frac{\gamma_*}{N}\sum_{i=1}^{N} n_i(t) - m$ we have that

$$\frac{1}{N}\sum_{i=1}^{N}\left(u_i^{(N)}(t) - \mu_1(t)\right)^2 \asymp \frac{1}{N}\sum_{i=1}^{N}\left(\frac{\gamma_*}{K}\sum_{j=1}^{N}\left(J_{ij}^{(N)} - \frac{K}{N}\right)n_i(t)\right)^2$$

$$= \left(\frac{\gamma_*}{K}\right)^2\sum_{j_1,j_2=1}^{N}\frac{1}{N}\sum_{i=1}^{N}\bar{J}_{ij_1}^{(N)}\bar{J}_{ij_2}^{(N)}n_{j_1}(t)n_{j_2}(t).$$

Again, using exchangeability, we obtain that **in expectation** the right hand side is equal to

$$\left(\frac{\gamma_*}{K}\right)^2\sum_{j=1}^{N}\frac{1}{N}\sum_{i=1}^{N}\left(\bar{J}_{ij}^{(N)}\right)^2 n_1(t)\left(1-n_2(t)\right) + \left(\frac{\gamma_*}{K}\right)^2\sum_{j=1}^{N}\frac{1}{N}\sum_{i=1}^{N}\left(\sum_{j=1}^{N}\bar{J}_{ij}^{(N)}\right)^2 n_1(t)n_2(t)$$

$$= \left(\frac{\gamma_*}{K}\right)^2\sum_{j=1}^{N}\frac{1}{N}\sum_{i=1}^{N}\left(\bar{J}_{ij}^{(N)}\right)^2 n_1(t)\left(1-n_2(t)\right)$$

$$= \left(\frac{\gamma_*}{K}\right)^2\frac{1}{N}\frac{K}{N}\left(1-\frac{K}{N}\right)n_1(t)\left(1-n_2(t)\right).$$

A formal replacement of $n_1(t)\,(1-n_2(t))$ with $m(t)(1-m(t))$ now yields

$$\frac{1}{N}\sum_{i=1}^{N}\left(u_i^{(N)}(t)-\mu_1(t)\right)^2 \asymp \frac{\gamma_*^2}{K}m(t)(1-m(t))\,.$$

The analysis of the asymptotic behaviour of the higher moments $k=3,4,\ldots$ soon becomes rather difficult. We can formally proceed as follows:

$$\frac{1}{N}\sum_{i=1}^{N}\left(u_i^{(N)}(t)-\mu_1(t)\right)^k = \left(\frac{\gamma_*}{K}\right)^k \frac{1}{N}\sum_{i=1}^{N}\left(\sum_{j=1}^{N} J_{ij}^{(N)} n_j(t) - Km(t)\right)^k\,.$$

To analyze the asymptotic behaviour of $\sum_{i=1}^{N}\left(\sum_{j=1}^{N} J_{ij}^{(N)} n_j(t) - Km(t)\right)^k$ note that

$$\sum_{j=1}^{N} J_{ij}^{(N)} n_j(t) = \sum_{l=1}^{K} n_{j_l}(t)$$

for certain $1\le j_1<j_2<\ldots<j_K\le N$, which has the same distribution as $\sum_{j=0}^{K-1} n_{i+j}(t)$ due to exchangeability. **In expectation** the right hand side is therefore equal to

$$\left(\frac{\gamma_*}{K}\right)^k \frac{1}{N}\sum_{i=1}^{N}\left(\sum_{j=0}^{K-1} n_{i+j}(t) - Km(t)\right)^k\,.$$

Assuming as before the asymptotic uncorrelatedness of $n_1,\ldots,n_N$, the law of large numbers now implies that

$$\lim_{N\to\infty}\frac{1}{N}\sum_{i=1}^{N}\left(\sum_{j=0}^{K-1} n_{i+j}(t) - Km(t)\right)^k = E\left((U(t)-Km(t))^k\right)$$

where $U(t)\sim \mathrm{Bin}(K,m(t))$.

Inserting the last asymptotics into the Taylor expansion yields the infinite series

$$\begin{aligned}\frac{1}{N}\sum_{i=1}^{N} f(u_i^{(N)}(s)) &\sim f(\mu_1(s)) + \sum_{k=2}^{\infty}\frac{f^{(k)}(\mu_1(s))}{k!}\left(\frac{\gamma_*}{K}\right)^k E\left((U(t)-Km(t))^k\right)\\ &= E\left(f\left(\mu_1(s)+\frac{\gamma_*}{K}(U(s)-Km(s))\right)\right) = E\left(f\left(\frac{\gamma_*}{K}U(s)-m\right)\right)\,.\end{aligned} \tag{8.3}$$

It follows that the mean activity in this setting should be well-described by the solution of the following ordinary differential equation

$$\dot m(t) = -m(t) + E\left(f\left(\frac{\gamma_*}{K}U(t)-m\right)\right) \text{ where } B(t)\sim\ \mathrm{Bin}(K,m(t))\,. \tag{8.4}$$

Note that in the case where $K\uparrow\infty$ the law of large numbers implies that the above expectation converges to $f(\gamma_* m(t)-m)$ and (8.4) reduces to the dynamical equation (8.2) for the mean activity obtained previously under the $K_N\uparrow\infty$ assumption.

to do: numerical comparison

It remains to prove the exchangeability of $n_1,\ldots,n_N$.

Lemma 8.5. *Assume that the joint distribution of $n_1(0), \dots, n_N(0)$ is exchangeable. Then the joint distribution of $n_1(t), \dots, n_N(t)$ is exchangeable too, i.e., for any permutation σ of $\{1, \dots, N\}$, and any bounded measurable function g*

$$E\left(g(n_{\sigma(1)}(t), \dots, n_{\sigma(N)}(t)) \mid (J_{ij})\right) = E\left(g(n_1(t), \dots, n_N(t)) \mid (J_{ij})\right) .$$

Proof. Fix g and apply the semimartingale decomposition to both $g(n(t))$ and $g(n_\sigma(t))$:

$$g(n(t)) = g(n(0)) + \int_0^t Qg(n(s))\, ds + M_t^g$$
$$g(n_\sigma(t)) = g(n_\sigma(0)) + \int_0^t Qg(n_\sigma(s))\, ds + M_t^{g_\sigma} .$$

This implies that

$$\begin{aligned}
&E\left(g(n(t)) - g(n_\sigma(t)) \mid (J_{ij})\right) = E\left(g(n(0)) - g(n_\sigma(0)) \mid (J_{ij})\right) \\
&\quad + \int_0^t \sum_{i=1}^N E\left((1-n_i(s))f(u_i^{(N)}(s))(g(n(s)+e_i) - g(n(s))) - (1-n_{\sigma(i)}(s))f(u_{\sigma(i)}^{(N)}(s))(g(n_\sigma(s)+e_{\sigma(i)}) - g(n_\sigma(s))) \mid (J_{ij})\right) ds \\
&\quad + \int_0^t \sum_{i=1}^N E\left(n_i(s)(1-f(u_i^{(N)}(s)))(g(n(s)-e_i) - g(n(s))) - n_{\sigma(i)}(s)(1-f(u_{\sigma(i)}^{(N)}(s)))(g(n_\sigma(s)-e_{\sigma(i)}) - g(n_\sigma(s))) \mid (J_{ij})\right) ds \\
&\quad = \int_0^t \sum_{i=1}^N E\left(h_i(n(s)) - h_i(n_\sigma(s)) \mid (J_{ij})\right) ds
\end{aligned}$$

with

$$h_i(n) = (1-n_i)f(u_i^{(N)})(g(n+e_i) - g(n)) + n_i(1-f(u_i^{(N)}))(g(n-e_i) - g(n)) .$$

Now define

$$d_N(t) := \sup_{\|g\|_\infty \le 1} \left|E\left(g(n(t)) - g(n_\sigma(t)) \mid (J_{ij})\right)\right| .$$

The previous computations then imply that

$$\left|E\left(g(n(t)) - g(n_\sigma(t)) \mid (J_{ij})\right)\right| \le \sum_{i=1}^N \|h_i\|_\infty \int_0^t d_N(s)\, ds$$

and taking sup over $\|g\|_\infty \le 1$, using $\|h_i\|_\infty \le 2\|g\|_\infty \le 2$, we conclude that $d_N(t) \le 2N \int_0^t d_N(s)\, ds$, hence $d_N(t) = 0$ which implies the assertion. □

Lemma 8.6. *(Joint distribution of (J_{ij})) Let $1 \le a_1 < \ldots < a_l \le N$. Then*

$$P\left(J_{ia_1} = 1, \dots, J_{ia_l} = 1\right) = \frac{(K)_l}{(N)_l} .$$

In particular,

$$P(J_{ij} = 1) = \frac{K}{N} \text{ and } P(J_{ij_1} = 1, J_{ij_2} = 1) = \frac{K(K-1)}{N(N-1)} \text{ for } j_1 \ne j_2 .$$

Proof. Note that

$$\begin{aligned}
P\left(J_{ia_1} = 1, \dots, J_{ia_l} = 1\right) &= \frac{\#\text{ of } (K-l)\text{-subsets of } \{1, \dots, N\} \setminus \{a_1, \dots, a_l\}}{\#\text{ of } K\text{-subsets of} \{1, \dots, N\}} \\
&= \binom{N-l}{K-l} \Big/ \binom{N}{K} = \frac{(K)_l}{(N)_l} .
\end{aligned}$$

□

8.2.4. Extensions to the case $K_N \equiv K$, $\gamma_N \equiv \gamma_*/\sqrt{K}$, K large. This is the most interesting regime, because it models typical neural activity in cortical circuits (see[**VS98**]). The input to neuron i in this scenario is now given as

$$u_i^{(N)}(t) = \frac{\gamma_*}{\sqrt{K}} \sum_{i=1}^{N} J_{ij}^{(N)} n_j(t) - \sqrt{K} m\,.$$

To understand the asymptotic behaviour of a typical input assume for the moment that the n_i are independent. In this case $u^{(N)}(t)$ models typical fluctuations of order $\sqrt{K}$ given for large K in terms of the classical central limit theorem. To generate a resting state, we lower the input just a little bit. In this state the network exhibits sparse activity, but a little amount of additional external input, e.g. a stimulus, can lead to large activity of the network.

To derive a differential equation for the mean activity we now just need to modify the formal computations from the previous scenario as follows, multiplying every moment in the Taylor expansion with $\sqrt{K}^k$, to get

$$\frac{1}{N} \sum_{i=1}^{N} f(u_i^{(N)}(s)) \sim E\left(f\left(\frac{\gamma_*}{\sqrt{K}} U(s) - \sqrt{K} m \right)\right) ,$$

where as before $U(s) \sim \mathrm{Bin}(K, m(s))$.

For increasing K, the asymptotics is now given in terms of the CLT-approximation $\mathrm{Bin}(K, m(t)) \sim N(Km(t), Km(t)(1-m(t)))$, hence

$$\frac{\gamma_*}{\sqrt{K}} U(s) - m \sim N\left((\sqrt{K}(\gamma_* m(s) - m), \gamma_*^2 m(s)(1-m(s)) \right) = N(\mu_1(s), \mu_2(s))$$

with $\mu_2(s) = \gamma_*^2 m(s)(1-m(s))$, which is K-times the previous $\mu_2(s)$. Now we obtain the integral representation

$$E\left(f\left(\frac{\gamma_*}{\sqrt{K}} U(s) - \sqrt{K} m \right)\right) \sim \frac{1}{\sqrt{2\pi \gamma_*^2 m(s)(1-m(s))}} \int f(u) e^{-\frac{(u-\sqrt{K}(\gamma_* m(s) - m))^2}{2\gamma_*^2 m(s)(1-m(s))}}\, du$$

obtained in [**VS98**].

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