Existence, uniqueness and stability of semi-linear rough partial differential equations

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Rough PDE via Feynman-Kac integrals

- Rough PDEs: motivation from stoch. filtering
- Rough linear PDE
- Rough semilinear PDE

Rough PDE via Davies' method

- Motivation: Burgers equation
- Variational approach
- Rough differential equations according to Davie
- A priori estimates
- Stability

Motivation: Stochastic filtering theory

Estimating partially observed diffusion process

$$
\begin{aligned}
&(\mathbf{S}) \qquad dX_t = b(X_t)\,dt + \sigma(X_t)\,dB_t \quad \in \quad \mathbb{R}^d \\
&(\mathbf{O}) \qquad dW_t = \gamma(X_t)\,dt + d\tilde{B}_t \quad \in \quad \mathbb{R}^e
\end{aligned}
$$

for (ind.) Brownian motions B, $\tilde{B}$

cond. expectation of (S) given $\mathcal{W}_{s:t} = \sigma\left(W_u : u \in [s,t]\right)$

$$
E^{t,x}\left[g(X_T) \mid \mathcal{W}_{t,T}\right] = \frac{E_0^{t,x}\left[g(X_T)\exp\left(\int_t^T \gamma(X_s)\,dW_s - \frac{1}{2}\int_t^T \|\gamma(X_s)\|^2\,ds\right)\right]}{E_0^{t,x}\left[\exp\left(\int_t^T \gamma(X_s)\,dW_s - \frac{1}{2}\int_t^T \|\gamma(X_s)\|^2\,ds\right)\right]}
$$

where $E_0^{t,x}$ denotes expectation w.r.t. B only (!)

Rem

- crucial pb: robustness w.r.t. data $W_{0:t}$, (in part. $\int_t^T \gamma(X_s)\,dW_s$)
- **our Ansatz** consider the rough path topology

Rough path integrals - main idea

- recall Young's ineq $\left|\int_s^t Y_r - Y_s \, dX_r\right| \le C_{\beta,\alpha}|t-s|^{\alpha+\beta} \, \|Y\|_\beta \|X\|_\alpha$, $\alpha+\beta > 1$

- but for $Y_t = F(X_t)$, $F \in C^2$

$$\int_s^t F(X_r)\, dX_r = F(X_s)(\underbrace{X_t - X_s}_{=:X_{st}}) + \int_s^t F(X_r) - F(X_s)\, dX_r$$

$$= F(X_s)X_{st} + DF(X_s) \underbrace{\int_s^t X_{sr}\, dX_r}_{=:\mathbb{X}_{st}} + R_{st}$$

with $\|R\|_{3\alpha} \le C\|D^2F\|_\infty \|X\|_\alpha^2$, the integral becomes continuous w.r.t. $(X_t, \mathbb{X}_{st})$ in $\|X\|_\alpha + \|\mathbb{X}\|_{2\alpha}$, sufficient for $\alpha > \frac{1}{3}$, since then

$$\int_s^t F(X_r)\, dX_r := \lim_{|\mathcal{P}|\to 0} \sum_{[u,v]\in\mathcal{P}} F(X_u)X_{uv} + DF(X_u)\mathbb{X}_{uv} \qquad \text{well-defined}$$

- admissible integrands: $Y \in C^\alpha$ s.th. $Y_{st} = Y'_s X_{st} + R_{st}$ with $Y' \in C^\alpha$ and $R \in C^{2\alpha}$

Back to Stochastic Filtering

$$E^{t,x}\left[g(X_T) \mid \mathcal{W}_{t,T}\right] = \frac{E_0^{t,x}\left[g(X_T)\exp\left(\int_t^T \gamma(X_s)\,dW_s - \frac{1}{2}\int_t^T \|\gamma(X_s)\|^2\,ds\right)\right]}{E_0^{t,x}\left[\exp\left(\int_t^T \gamma(X_s)\,dW_s - \frac{1}{2}\int_t^T \|\gamma(X_s)\|^2\,ds\right)\right]}$$
$$= \frac{u(t,x)}{u_0(t,x)}$$

where

$$u(t,x) = g(x) + \int_t^T Lu(r,x)\,dr + \int_t^T \gamma_k u(r,x)\,dW_r^k$$

(and u_0 the same backward eq with terminal condition $g(x) \equiv 1$)
with

$$Lu := \frac{1}{2}\mathrm{Tr}\left[\sigma\sigma^T(x)D^2u\right] + \langle b(x), Du\rangle$$

formally differentiating W^k yields the (backward rough) PDE

$$\begin{cases} -\partial_t u(t,x) & = Lu(t,x) + \gamma_k(x)u(t,x)\dot{W}_t^k \\ u(T,x) & = g(x),\, t \in [0,T],\, x \in \mathbb{R}^d. \end{cases}$$

Rough linear PDE

$$\begin{cases} \partial_t u(t,x) & = Lu(t,x) + \Gamma_k u(t,x) \dot{W}_t^k \\ u(0,x) & = g(x), t \in [0,T], x \in \mathbb{R}^d . \end{cases} \quad (1)$$

where $\mathbf{W}$ is a (geometric) α-Hölder rough path, $\alpha \in (1/3, 1/2]$ and

$$Lu := \frac{1}{2} \mathrm{Tr} \left[\sigma \sigma^T (x) D^2 u \right] + \langle b(x), Du \rangle + c(x) u(x)$$

$$\Gamma_k u := \langle \beta_k (x), Du \rangle + \gamma_k (x) u$$

Ansatz in [Friz, Diehl, S. 2017]: (rough path) Feynman-Kac representation

$$u^{\mathbf{W}}(t,x) = \mathbb{E}^x \left[g(X_t) \exp \left(\int_0^t c(X_r) \, dr + \int_0^t \gamma(X_r) \dot{W}_r dr \right) \right] \quad (2)$$

with

$$dX_t = \sigma(X_t) dB_t + b(X_t) dt + \beta(X_t) \dot{W}_t dt$$

where B is a Brownian motion

Rough linear PDE, ctd.

$$\begin{cases} \partial_t u(t,x) & = Lu(t,x) + \Gamma_k u(t,x) \dot{W}_t^k \\ u(0,x) & = g(x), t \in [0,T], x \in \mathbb{R}^d. \end{cases} \tag{1}$$

where $\mathbf{W}$ is a (geometric) α-Hölder rough path, $\alpha \in (1/3, 1/2]$ and

$$Lu := \frac{1}{2} \mathrm{Tr} \left[\sigma \sigma^T (x) D^2 u \right] + \langle b(x), Du \rangle + c(x) u(x)$$
$$\Gamma_k u := \langle \beta_k(x), Du \rangle + \gamma_k(x) u$$

Ansatz in [Friz, Diehl, S. 2017]: understand how

$$u^{\mathbf{W}}(t,x) = \mathbb{E}^x \left[g(X_t) \exp \left(\int_0^t c(X_r)\, dr + \int_0^t \gamma(X_r)\, \dot{W}_r dr \right) \right] \tag{2}$$

depends on $\mathbf{W} = (W, \mathbb{W})$ in the **rough path metric**,

reduces to understand stability of $\mathbf{W} \mapsto X$, where

$$dX_t = \sigma(X_t)\, dB_t + b(X_t)\, dt + \beta(X_t)\, \dot{W}_t dt$$

Rough linear PDE: weak solutions

Theorem

[Friz, Nilssen, S. 2018]

Assume $\sigma_{i,k}, \beta_j \in C_b^3(\mathbb{R}^d)$, $b_j, \gamma_j, c \in C_b^1(\mathbb{R}^d)$. Given $g \in L^p(\mathbb{R}^d)$, the Feynman-Kac integral (2) yields an analytically weak solution u of (1) satisfying

$$\sup_{t\in[0,T]} \|u(t)\|_{L^p} \leq C\|g\|_{L^p}$$

where C depends on $\mathbf{W}$ only through the rough path metric.

Rem In the simple case $b = c = \gamma = 0$

- $C = \mathbb{E}\left[\sup_{x\in\mathbb{R}} |\det D\Phi_{0,t}^{-1}(x)|\right]$
- Φ being the flow generated by $dX_t = \sigma(X_t)\,dB_t + \beta(X_t)\,d\mathbf{W}_t$

Rough linear PDE: regular solutions

Theorem

[Friz, Nilssen, S. 2018]

Assume $\sigma_{i,k}, \beta_j, \gamma_j \in C_b^6(\mathbb{R}^d)$, $b_j, c \in C_b^4(\mathbb{R}^d)$. Given $g \in W^{3,p}(\mathbb{R}^d)$, the Feynman-Kac integral (2) yields a weak solution $u \in W^{3,p}$ of (1) satisfying

$$\sup_{t\in[0,T]} \|u(t)\|_{W^{3,p}} \leq C\|g\|_{W^{3,p}}$$

where C depends on $\mathbf{W}$ only through the rough path metric.

Rem In this case $x \mapsto \Phi_{0,t}(x)$ is $C_b^3(\mathbb{R})$.

Rough semilinear PDE

Extensions to the semilinear case

$$\begin{cases} \partial_t u(t,x) & = Lu(t,x) + F(u)(t,x) + \Gamma_k u(t,x) \dot{W}_t^k \\ u(0,x) & = g(x)\,,\, t \in [0,T]\,,\, x \in \mathbb{R}^d\,. \end{cases} \tag{3}$$

using Duhamels principle

$$u(t) = P^{\mathbf{W}}_{0,t} g + \int_0^t P^{\mathbf{W}}_{s,t} F(u(s))\, ds \qquad \text{(e.g.) in } L^2$$

where $P^{\mathbf{W}}_{s,t}$ is the L^2-propagator of the linear Pb

requires a priori energy estimates of the type

- $\|u(t)\|_{L^2} \le C\|g\|_{L^2}$
- $\int_0^t \|u(r)\|^2_{W^{1,2}}\, dr \le C\|g\|^2_{L^2}$

Linear rough PDE: energy estimates

Ansatz Consider the rough heat equation

$$\begin{cases} \partial_t u(t,x) & = \Delta u(t,x) + \Gamma_k u(t,x) \dot{W}_t^k \\ u(0,x) & = g(x)\,, t \in [0,T]\,, x \in \mathbb{R}^d\,. \end{cases}$$

for smooth $\mathbf{W}$

usual integration by parts leads to

$$\|u(t)\|_{L^2}^2 + 2\int_0^t \|\nabla u(r)\|_{L^2}^2\, dr = \|u(0)\|_{L^2}^2 + 2\int_0^t (u(r)^2, \Gamma_k^* 1)\, dW_r^k$$

major problem: the elementary control of the 'rough integral'

$$\int_0^t (u(r)^2, \Gamma_k^* 1) dW_r^k\, dr \leq \|\Gamma_k^* 1\|_{L^\infty} \int_0^t \|u(r)\|_{L^2}^2\, d|W_r^k|$$

is not continuous in the rough path metric

Rough linear PDE: energy estimates, ctd.

Introducing the backward rough pde

$$\partial_t \varphi^{\mathbf{W}}(t,x) + \Delta \varphi^{\mathbf{W}}(t,x) + \Gamma_k^* \nabla \varphi^{\mathbf{W}}(t,x) \dot{W}_t^k + \Gamma_k^* 1(x) \varphi^{\mathbf{W}}(t,x) \dot{W}_t^k = 0$$

with nonnegative terminal condition, e.g. $\varphi^{\mathbf{W}}(T,x) \equiv 1$ leads to

$$\int u(t)^2 \varphi^{\mathbf{W}}(t)\, dx + 2 \int_0^t \int |\nabla u(r)|^2 \varphi^{\mathbf{W}}(r)\, dx\, dr = \int u(0)^2 \varphi^{\mathbf{W}}(0)\, dx \qquad t \in [0,T]$$

yields a priori energy estimate

as long as $\mathbf{W} \mapsto \varphi^{\mathbf{W}}(t)$ is cont. in the rough path metric
(e.g. via Feynman-Kac representation)

Rough semilinear PDEs - main result

Theorem

[Friz, Nilssen, S. 2018]

Assume $\sigma_{i,k}, \beta_j, \gamma_j \in C_b^6(\mathbb{R}^d)$, $b_j, c \in C_b^4(\mathbb{R}^d)$ as well as

- $\lambda|\xi|^2 \leq |\sigma^T \xi|^2$ for some $\lambda > 0$,
- $\|F(u) - F(v)\|_{L^2} \leq C\|u - v\|_{W^{1,2}(\mathbb{R}^d)}$.

Given $g \in L^2(\mathbb{R}^d)$, there exists a unique weak solution $u \in C([0,T]; L^2) \cap L^2([0,T]; W^{1,2})$ of (3).

Burgers equation

Navier Stokes equations

$$\partial_t u + u \cdot \nabla u + \nabla p = \nu \Delta u, \qquad (t,x) \in [0,T] \times \mathbb{R}^d$$
$$\text{div } u = 0, \quad u(0) = u_0 \in L^2(\mathbb{R}^d; \mathbb{R}^d).$$

have no meaning when $d = 1$

`[Burgers, 1974]` introduced the 1d toy-equation

$$\partial_t u + u \partial_x u = \nu \partial_x^2 u, \qquad (t,x) \in [0,T] \times \mathbb{R}$$

neglecting the pressure in the Navier-Stokes equations.

Stochastic Burgers equation

Adding randomness to the equation to incorporate

- fluctuations in classical continuum limits
- incorporate highly oscillating terms on small time-scales as statistical model for turbulence

first approach: additive, space-time white noise, ξ

$$\partial_t u = \nu \partial_x^2 u - u \partial_x u + \xi,$$

- [Bertini, Cancrini, Jona-Lasinio 1994]
- [DaPrato, Debussche, Temam 1994] (1st step towards regularity structures)

Stochastic Burgers equation with transport noise

Lagrangian viewpoint

$$\phi_t(x) := \text{ position of a fluid particle at time } t \text{ starting in } x$$

assume the following decomposition

$$\dot{\phi}_t(x) = u(t, \phi_t(x)) - \beta_j(\phi_t(x))\dot{Z}^j_t, \quad \phi_0(x) = x,$$

with $(Z^j)_j$ modelling a highly oscillating part

associated velocity field u is generated by

$$\partial_t u = \nu \partial_x^2 u - u \partial_x u + \beta_j \partial_x u \dot{Z}^j \tag{4}$$

Ansatz [Hocquet, Nilssen, S. 2018] prove stability of

$$(Z, \mathbb{Z}) \mapsto u \qquad \text{w.r.t. rough path metric}$$

Variational approach

consider the weak formulation

$$\partial_t \int u_t \phi \, dx = -\int \nabla u_t \cdot \nabla \phi \, dx - \int u_t div(\beta_j \phi) \, dx \, \dot{Z}_t^j$$

for all $\phi \in W^{1,2}(\mathbb{R})$ w.r.t. the Gelfand triple

$$W^{1,2}(\mathbb{R}) \subset L^2(\mathbb{R}) \subset W^{-1,2}(\mathbb{R})$$

testing u against itself yields the energy estimate

$$\begin{aligned}\|u_t\|_{L^2}^2 + 2\int_0^t \|\nabla u_r\|_{L^2}^2 dr &= \|u_0\|_{L^2}^2 - \int_0^t (u_r^2, div(\beta_j))dZ_r^j \\ &\leq \|u_0\|_{L^2}^2 + \|div(\beta_j)\|_{L^\infty} \int_0^t \|u_r\|_{L^2}^2 d|Z^j|_r\end{aligned}$$

because of $\int u_t \partial_x u_t u_t \, dx = -\frac{1}{3}\int \partial_x(u_t^3)\, dx = 0$

Rem

- again, no easy a priori (energy) estimates
- but also no easy perturbative approach

Davies' method for solving rough ode's

main idea consider the rough ode

$$dy_t = g(y_t)dX_t,$$

a Taylor expansion yields

$$y_{st} := y_t - y_s = g(y_s)X_{st} + \nabla g(y_s)g(y_s)\mathbb{X}_{st} + y^\natural_{st},$$

Definition y is called a solution if $y^\natural$ is more regular in time in the sense that

$$|y^\natural_{st}| \lesssim |t-s|^{1+}$$

Davies' method for solving rough pde's

consider pure transport eqn (and a smooth path Z)

$$\partial_t f = (\beta_j \cdot \nabla) f \dot{Z}_t^j, \qquad f|_{t=0} \in L^2(\mathbb{R}^d)$$

a similar Taylor expansion now yields

$$f_{st} = (\beta_j \cdot \nabla) f_s Z_{st}^j + (\beta_j \cdot \nabla)(\beta_k \cdot \nabla) f_s \mathbb{Z}_{st}^{k,j} + f_{st}^\natural$$

where

$$f_{st}^\natural(x) = \int_s^t \int_s^{r_1} \int_s^{r_2} (\beta_j \cdot \nabla)(\beta_k \cdot \nabla)(\beta_i \cdot \nabla) f_{r_3}(x) dZ_{r_3}^i dZ_{r_2}^k dZ_{r_1}^j$$

Rem requires 3 derivatives of f

Davies' method for solving rough pde's, ctd.

for the variational approach therefore need

$$W^{3,2}(\mathbb{R}^d) \subset L^2(\mathbb{R}^d) \subset W^{-3,2}(\mathbb{R}^d)$$

define the unbounded rough drivers

$$A^1_{st}\phi := (\beta_j \cdot \nabla)\phi Z^j_{st} \qquad A^2_{st}\phi := (\beta_j \cdot \nabla)(\beta_k \cdot \nabla)\phi \mathbb{Z}^{k,j}_{st}$$

satisfying

$$|A^1_{st}|_{\mathcal{L}(W^n, W^{n-1})} \lesssim |t-s|^{\alpha} \qquad |A^2_{st}|_{\mathcal{L}(W^n, W^{n-2})} \lesssim |t-s|^{2\alpha}$$

and Chen's relation $A^2_{st} - A^2_{s\theta} - A^2_{\theta t} = A^1_{\theta t} A^1_{s\theta}$

Definition $f : [0, T] \to L^2(\mathbb{R}^d)$ is called a solution if

$$f_{st} = A^1_{st} f_s + A^2_{st} f_s + f^\natural_{st}$$

with

$$|f^\natural_{st}(\phi)| \lesssim |t-s|^{1+} |\phi|_{W^3}.$$

A priori estimates

consider the equation perturbed by a drift

$$f_{st} = \mu_{st} + A^1_{st} f_s + A^2_{st} f_s + f^\natural_{st}$$

where $\mu : [0, T] \to W^{-2,2}(\mathbb{R}^d)$ is Lipschitz, e.g. $\mu = \Delta f$

Proposition

`[Deya, Gubinelli, Hofmanova, Tindel 2016]`

Suppose we have a solution to the above equation. Then

$$|f^\natural_{st}(\phi)| \lesssim (\mu_{Lip} \|A\|_\alpha + |f|_{L^{\infty,2}_{T,x}} \|A\|^2_\alpha) |\phi|_{W^3} |t-s|^{1+}.$$

Stability

to obtain stability of

$$v_{st} = \int_s^t \Delta v_r dr + A^1_{st} v_s + A^2_{st} v_s + v^\natural_{st}$$

in the rough path topology we now need 3 derivatives

using a tensorization argument and involved approximation by smooth solutions we obtain

$$v^2_{st}(\phi) = -2\int_s^t |\nabla v_r|^2(\phi) + \nabla v_r (v_r \nabla \phi) dr$$
$$+ v^2_s(A^{1,*}_{st}\phi) + v^2_s(A^{2,*}_{st}\phi) + v^{2,\natural}_{st}(\phi)$$

with the a priori estimate

$$|v^{2,\natural}_{st}(\phi)| \lesssim_A \left(\|\nabla v\|^2_{L^2([s,t]\times\mathbb{R}^d)} + \|v\|^2_{L^\infty([s,t];L^2(D))} \right) \|\phi\|_{W^{3,\infty}} |t-s|^{1+}$$

Stability, ctd.

to obtain stability of

$$v_{st} = \int_s^t \Delta v_r \, dr + A^1_{st} v_s + A^2_{st} v_s + v^\natural_{st}$$

in the rough path topology we now need 3 derivatives

letting $\phi = 1$ we get

$$\begin{aligned}(\|v\|^2_{L^2})_{st} + 2\int_s^t \|\nabla v_r\|^2_{L^2} dr &= v^2_s(A^{1,*}_{st} 1) + v^2_s(A^{2,*}_{st} 1) + v^{2,\natural}_{st}(1) \\ &\lesssim \|A\|_\alpha |t-s|^\alpha \big(\|\nabla v\|^2_{L^2([s,t]\times\mathbb{R}^d)} + \|v\|^2_{L^\infty([s,t];L^2(\mathbb{R}^d))}\big)\end{aligned}$$

(rough) Gronwall lemma $\Rightarrow$ energy estimate

Application to Rough Burgers

$$\partial_t u = \partial_x^2 u - u\partial_x u + \beta_j \partial_x u \dot{Z}_t^j$$

formally yields the following eqn for u^2

$$\partial_t u^2 = 2u\partial_x^2 u - 2u^2 \partial_x u + \beta_j \partial_x u^2 \dot{Z}_t^j$$

the remainder in this case can be estimated by

$$|u_{st}^{2,\natural}(\phi)| \lesssim_A \left(\|\nabla u\|^2_{L^2([s,t];L^2)} + \|u\|^2_{L^\infty([s,t];L^2)} + \|u\|^3_{L^\infty([s,t];L^3)}\right) \cdot \\ \cdot \|\phi\|_{W^{3,\infty}} |t-s|^{1+}$$

Rem cubic in u, so rough Gronwall lemma not applicable

Weighted measure space approach

similar to part 1 introduce the backward rough pde

$$\partial_t \varphi_t^Z = -\partial_x^2 \varphi_t^Z + \partial_x(\beta_j \varphi_t^Z)\dot{Z}_t^j, \quad \varphi_T^Z \equiv 1$$

leads to the weighted energy equality

$$\partial_t \int u_t^2 \varphi_t^Z \, dx + 2 \int (\partial_x u_t)^2 \varphi_t^Z \, dx = 2 \int u_t^2 \partial_x u_t \varphi_t^Z \, dx$$

if we can show that $0 < m_* \leq \varphi_t^Z(x) \leq m^* < \infty$

$$\|u_T\|_{L^2}^2 + \int_0^T \|\partial_x u_t\|_{L^2}^2 dt$$

$$\lesssim_{m_*, m^*} \|u_0\|_{L^2}^2 + 2 \int_0^T (|u_t|^2, |\partial_x u_t|) dt$$

still cubic in u on RHS

Use non-linear Gronwall:

Theorem (Bihari-LaSalle)

Suppose $q > 1$ and two positive functions $x, k : [0, T] \to \mathbb{R}_+$ satisfy

$$x_t \leq x_0 + \int_0^t k_s x_s^q ds.$$

Then for all t such that $(q-1)x_0 \int_0^t k_s ds < 1$ we have the following estimate

$$x_t \leq \frac{x_0}{\left(1 - (q-1)x_0 \int_0^t k_s ds\right)^{1/(q-1)}}.$$

Backward equation

main advantage of the backward equation

$$\partial_t \varphi_t^Z = -\partial_x^2 \varphi_t^Z + \partial_x(\beta_j \varphi_t^Z)\dot{Z}_t^j, \quad m_T = 1$$

admits the Feynman-Kac representation

$$\varphi_t^Z(x) = E\left[\exp\left(\int_t^T (\partial_x \beta_j)(\psi_{t,s}(x))dZ_s^j\right)\right]$$

where

$$d\psi_{t,s}(x) = \sqrt{2}dB_s + \beta_j(\psi_{t,s}(x))dZ_s^j, \quad \psi_{t,t}(x) = x$$

So in fact this method can be seen as a flow-transformation on the energy space

Main result

This gives

$$\partial_t(u_t^2, m_t) + 2(|\partial_x u_t|^2, m_t) = 2(u_t^2, \partial_x(u_t m_t)).$$

Upper and lower bound on m can be checked directly from the Feynman-Kac representation.

Theorem

`[Hocquet, Nilssen, S. 2018]`

There exists a maximal time $T_0 \in (0, T]$ such that existence and uniqueness of finite-energy solutions u to

$$\partial_t u = \partial_x^2 u - u\partial_x u - \beta_j \partial_x u \dot{Z}^j$$

holds on $(0, T_0) \times \mathbb{R}$
The value of T_0 depends only upon the quantities $\|u_0\|_{L^2}$, $\|\beta_j\|_{C_b^6}$, $\|Z\|_\alpha$, $\|\mathbb{Z}\|_{2\alpha}$