Stochastic Navier-Stokes Equations with Coriolis force

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  (joint with M. Hieber (TU Darmstadt),
  to appear in Ann. Scuola Norm. Sup. Pisa)
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  (partly joint with A. Es Sarhir (Agadir))

Navier-Stokes equation with Coriolis term

bounded layer $\mathbb{R}^2 \times (0,b),\ b > 0$

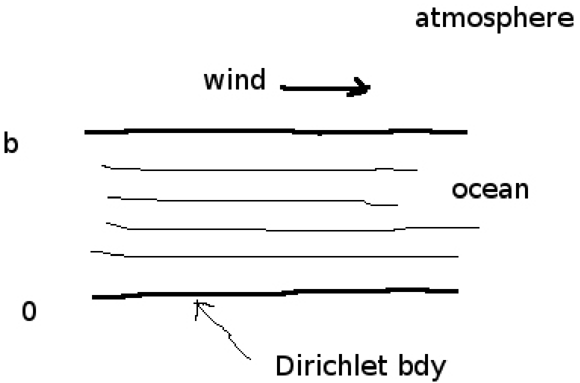
$$
\left\{
\begin{aligned}
\partial_t u - \nu \Delta u + 2\omega(e_3 \times u) + (u \cdot \nabla)u + \nabla p &= 0 \\
div\ u &= 0 \\
u(t, x_1, x_2, 0) &= 0 \\
u(t, x_1, x_2, b) &= u_b\, e_1 \\
u(0, \cdot) &= u_0
\end{aligned}
\right. \tag{1}
$$

where

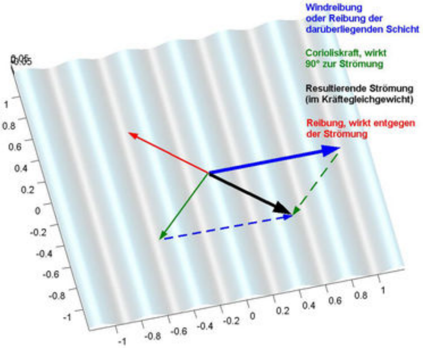
- u - velocity
- p - pressure
- $\nu > 0$ viscosity
- $\omega \in \mathbb{R}$ angular velocity
- $u_b e_1,\ u_b \in \mathbb{R}$ surface velocity

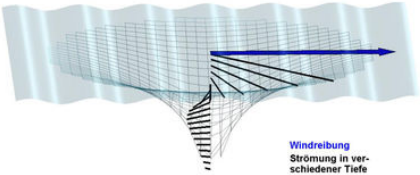
describes motion of rotating fluids influenced by Coriolis force

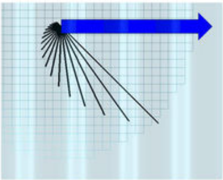
Illustration



Ekman spiral







(**from** Wikipedia)

Ekman spiral - analytic expression

explicit stationary solution of (1)

$$u_b^E(x_1, x_2, x_3) = \tilde{u}_b \begin{pmatrix} 1 - e^{-x_{3/\delta}} \cos(x_{3/\delta}) \\ e^{-x_{3/\delta}} \sin(x_{3/\delta}) \\ 0 \end{pmatrix}$$

$$p_b^E(x_1, x_2, x_3) = -\omega \tilde{u}_b x_2$$

with $\delta = \frac{b}{k\pi}$, $k \in \mathbb{Z}$, and

$$\tilde{u}_b = \begin{pmatrix} u_b(1 - e^{-\frac{b}{\delta}})^{-1} & \text{if } k \text{ is even} \\ u_b(1 + e^{-\frac{b}{\delta}})^{-1} & \text{if } k \text{ is odd.} \end{pmatrix}$$

Background: mathematical motivation

- (1) has a limiting equation for $|\omega| \to \infty$
- $|\omega| \gg 1$ implies
  - ex. of global strong solutions
    (Babin, Mahalov, Nicolaenko 1999)
  - $|\omega| \gg 1$ ex. and uniqueness of mild solutions
    (e.g.: Chemin, et. al.: Math. Analysis of Rotating Fluids, 2006)

for arbitrary initial data u_0

Known results: existence, stability

Function spaces

for $v = u - u_b^E$

$$\mathcal{D} := \left\{ v \in C_0^\infty\left(\mathbb{R}^2 \times (0,b)\right)^3 : \ \mathrm{div}\ v = 0,\ v(x_1,x_2,0) = 0 = v(x_1,x_2,b) \right\}$$

$$H := \overline{\mathcal{D}}^{L^2\left(\mathbb{R}^2\times(0,b)\right)^3} \qquad \text{and} \qquad V := \overline{\mathcal{D}}^{H^{1,2}\left(\mathbb{R}^2\times(0,b)\right)^3}$$

Known results, ctd.

Hess, Hieber, Mahalov, Saal (2010)

- existence of global weak solutions

$$u = v + u_b^E\,, \quad v \in L^\infty(0, T; H) \cap L^2(0, T; V)$$

- if

$$\lambda_0 := \frac{\pi^2}{b^2}\left(\nu - \sqrt{2}\tilde{u}_b\left(\delta - (b+\delta)e^{-\frac{b}{\delta}}\right)\right) \geq 0$$

  then $\exists$ (at least one) global weak solution $u = v + u_b^E$ such that $\lim_{t\to\infty} \|v(t)\|_H = 0$, in addition

$$\|v(t)\|_H \leq e^{-\lambda t}\|v_0\|_H \text{ for any } \lambda < \lambda_0$$

Stochastic Navier-Stokes-Coriolis equations

restricting to (x_1, x_2)-periodic solutions u

$\Omega_b := \mathbb{T}^2 \times (0, b)$, $\mathbb{T}^2$ 2D-Torus

Function spaces

for $v = u - u_b^E$

$$\mathcal{D} := \left\{ v \in C^\infty(\Omega_b)^3 : \text{ div } v = 0, v(x_1, x_2, 0) = 0 = v(x_1, x_2, b) \right\}$$

$$H := \overline{\mathcal{D}}^{L^2(\Omega_b)^3} \qquad \text{and} \qquad V := \overline{\mathcal{D}}^{H^{1,2}(\Omega_b)^3}$$

$$\begin{cases} du_t = [\nu \Delta u_t - 2\omega(e_3 \times u_t) - (u_t \cdot \nabla) u_t + \nabla p_t]\, dt + dW_t \\ div\ u_t = 0 \\ u_t(x_1, x_2, 0) = 0 \\ u_t(x_1, x_2, b) = u_b\, e_1 \end{cases} \tag{2}$$

$(W_t)_{t \geq 0}$ Wiener process on H

Stochastic evolution equation

orthogonal-projection $\Pi : L^2(\Omega_b)^3 \to H$, applied to (2), yields

$$du_t = [\nu A_S u_t - 2\omega\Pi(e_3 \times u_t) - \Pi(u_t \cdot \nabla)u_t]\, dt + dW_t \qquad (3)$$

on the space H

$$A_S u = \nu\, \Pi\Delta u \qquad \text{Stokes operator}$$

decompose

$$u_t = v_t + u_b^E \qquad \text{w.r.t. the Ekman spiral } u_b^E$$

then v solves

$$dv_t = [(\nu A_S + B)v_t - \Pi(v_t \cdot \nabla)v_t]\, dt + dW_t$$

where

$$B^{u_b^E} v := -2\omega\, \Pi(e_3 \times v) - \Pi(u_b^E \cdot \nabla v) - \Pi(v_3 \partial_3 u_b^E)$$

Perturbed stoch. Navier-Stokes eq

$$dv_t = [(\nu A_S + B)v_t - \Pi(v_t \cdot \nabla)v_t]\, dt + dW_t \qquad (4)$$

Assumptions

A.1 $B : V \to H$, bounded, linear, s.th. $\exists\, \omega_0 > 0$, $\omega_1 \geq 0$ with

$$\langle (\nu A_S + B)u, u\rangle_H \leq -\omega_0 \|u\|_V^2 + \omega_1 \|u\|_H^2 \qquad \forall u \in D(A_S)\,.$$

A.2 $(W_t)_{t\geq 0}$ is a Wiener-process on H with covariance Q having finite trace

Rem A.1 implies that $(\nu A_S + B, D(A_S))$ generates an analytic C_0-semigroup $(T_t)_{t\geq 0}$ satisfying $\|T_t\|_{L(H)} \leq e^{(\omega_1 - \tilde{\omega}_0)t}$, $t \geq 0$, $\tilde{\omega}_0 = \omega_0 \frac{\pi^2}{b^2}$

Weak martingale solutions

Theorem (S., Hieber 2009) Let $\xi : \Omega \to H$ be $\mathcal{F}_0$-measurable, in $\mathcal{L}^2(P)$, independent of $(W_t)_{t \geq 0}$.

Then $\exists$ weak martingale solution $((v_t), (\tilde{W}_t))$ of (4) satisfying $\tilde{P} \circ v_0^{-1} = P \circ \xi^{-1}$

Moreover

$$\tilde{E}\left(\sup_{0 \leq t \leq T} \|v_t\|_H^2 + \int_0^T \|v_t\|_V^2 \, dt\right) < \infty$$

proof similar to the unperturbed case (e.g. Flandoli, Gatarek [PTRF 1995]) via Galerkin approximation, tightness, ...

Corollary $B = B^{u_b^E}$, then $u_t := v_t + u_b^E$ is a weak martingale solution of (3).

Invariant measures

consider again the perturbed stochastic Navier-Stokes equation (4)

assume

$$\langle (\nu A_S + B)u, u \rangle_H \leq -\tilde{\omega}_0 \|u\|_V^2 \quad \forall u \in D(A_S)$$

with $\tilde{\omega}_0 := \omega_0 - \frac{1}{\nu}\frac{b^2}{\pi^2}\omega_1 > 0$

Theorem (S., Hieber, 2009) $\exists$ a stat. martingale solution of (4) Moreover, the invariant distribution $\mu = \tilde{P} \circ v_t^{-1}$ satisfies the moment estimate

$$\int \left(1 + \|x\|_V^2\right) e^{\varepsilon \|x\|_H^2} \mu(dx) < \infty \qquad \text{for } \varepsilon < \varepsilon_0 := \frac{\tilde{\omega}_0}{\|Q\|_{L(H)}}$$

Navier-Stokes-Coriolis eq on $\mathbb{T}^2$

appropriate projection of (1) onto (x_1, x_2)-plane

$$\begin{cases} \partial_t u - \nu\Delta u + \ell e_3 \times u - (u \cdot \nabla)u + \nabla p = 0 \\ \operatorname{div} u = 0 \\ u(0, \cdot) = u_0 \end{cases} \tag{5}$$

where

- $\ell = \omega + \beta x_2$
- $$\ell e_3 \times u = \omega \begin{bmatrix} -u_2 \\ u_1 \end{bmatrix} + \beta x_2 \begin{bmatrix} -u_2 \\ u_1 \end{bmatrix}$$

ex. & uniq. well-known

Stoch. Navier-Stokes-Coriolis eq on $\mathbb{T}^2$

$$\begin{cases} du_t = [\nu \Delta u_t - \ell(e_3 \times u_t) - (u_t \cdot \nabla) u_t + \nabla p_t]\, dt + dW_t \\ \qquad\qquad div\, u_t = 0 \end{cases} \tag{6}$$

$(W_t)_{t\geq 0}$ Wiener process on $H := \overline{\mathcal{D}}^{L^2(\mathbb{T}^2)^2}$
where

$$\mathcal{D} := \left\{ u \in C^\infty \left(\mathbb{T}^2\right)^2 : \text{ div } u = 0, \int_{T^2} u\, dx = 0 \right\}$$

Stoch. Navier-Stokes-Coriolis eq on $\mathbb{T}^2$, ctd.

applying the Helmholtz/Leray-projection yields

$$du_t = [\nu A_S u_t + Bu_t - \Pi (u_t \cdot \nabla) u_t] \, dt + dW_t \tag{7}$$

where

$$Bu = \Pi \left(\omega \begin{bmatrix} -u_2 \\ u_1 \end{bmatrix} + \beta x_2 \begin{bmatrix} -u_2 \\ u_1 \end{bmatrix} \right) = \beta \Pi \left(x_2 \begin{bmatrix} -u_2 \\ u_1 \end{bmatrix} \right)$$

Main features of B

- $\langle Bu, u \rangle = 0$ (cons. of energy)
- $\langle Bu, A_S u \rangle = 0$ (cons. of enstrophy)

Rem ex. of (stationary) weak martingale solution (should be) similar to 3D (not done)

Kolmogorov operator

instead Cauchy-problem

$$\frac{d}{dt} F = LF \, , F(0, \cdot) = F_0 \cdot \tag{8}$$

for the ass. Kolmogorov operator

$$LF(u) := \frac{1}{2} \operatorname{tr} \left(Q D^2 F(u) \right) + \langle \nu A_S u + Bu - \Pi(u \cdot \nabla) u, DF(u) \rangle$$

that gives the (unique?) transition semigroup ass. with (7)

$$F(t, u) = E \left(F_0(u_t) \mid u_0 = u \right)$$

main difficulty L only explicit on

$$\mathcal{F}C_b^2 := \{ F(u) = \varphi(\langle e_1, u \rangle, \ldots, \langle e_n, u \rangle) : n \geq 1, \\ e_1, \ldots, e_n \in D(A_S), \varphi \in C_b^2(\mathbb{R}^n) \}$$

L^p-analysis of Kolmogorov operators

a nice theory for (8) can be established in $L^p(\mu)$, where μ is infinitesimally invariant for L, i.e.,

$$L(\mathcal{F}C_b^2) \subset L^1(\mu) \qquad \text{and} \qquad \int LF\, d\mu = 0 \quad \forall F \in \mathcal{F}C_b^2$$

Implications

- Beurling-Deny criteria ("integr. maximum principle")

$$\int LF\Psi(F)\, d\mu \leq 0 \qquad \forall\, \Psi \in C(\mathbb{R}), \uparrow$$

- $(L, \mathcal{F}C_b^2(D(A)))$ is **dissipative** on $L^p(\mu)$ $\forall\, p \in [1, \infty[$
- L L^p-unique iff

  (i) the closure in $L^p(\mu)$ generates C_0-semigroup $(T_t^p)_{t\geq 0}$
  (ii) $\exists \lambda > 0$ with $(\lambda - L)(\mathcal{F}C_b^2(D(A))) \subset L^p(\mu)$ dense
  range condition

- in this case: $(T_t^p)_{t\geq 0}$ **Markovian**
  and stationary martingale solution of (7) unique

Program worked out for general semilinear SPDE

(joint with Es-Sarhir)

$$du_t = (Au_t + B(u_t))\, dt + C\, dW_t \in H$$

where

- A self-adjoint, of negative type, compact resolvent
- B loc. Lipschitz on $D((-A)^\gamma)$, $\gamma \in [0, \frac{1}{2}[$, (JEE 2008) (e.g. stochastic reaction diffusion equations)
- B one-sided Lipschitz (JDE 2009) (e.g. stochastic Cahn-Hilliard equation)
- $B = B_0 + B_1$, B_0 one-sided Lipschitz, $|B_1|_{D((-A)^\gamma)} \in L^2(\mu)$ (JFA 2010) (e.g. stochastic Burgers equation)

Back to 2D-NSC: Invariant measures - finite trace

Theorem (S. Math. Nachr. 2011)

- $\mathrm{tr}_H(Q) < \infty \Longrightarrow \exists\ \mu$ inf. invariant for L with

$$\int \left(1 + \|u\|_V^2\right) e^{\varepsilon \|u\|_H^2}\, \mu(dx) < \infty$$

  if $\varepsilon < \frac{\nu}{\|Q\|_{L(H)}}$

- $\mathrm{tr}_V(Q) < \infty \Longrightarrow \mu$ satisfies in addition

$$\int \left(1 + \|u\|_{H^2}^2\right) e^{\varepsilon \|u\|_V^2}\, \mu(dx) < \infty \qquad (9)$$

  if $\varepsilon < \frac{\nu}{\|Q\|_{L(V)}}$

L^p-uniqueness - finite trace

Theorem (S. Math. Nachr. 2011)
Let μ be inf. invariant, satisfying (9), then

$(L, \mathcal{F}C_b^2)$ is L^p-unique for all $p \in [1, \infty[$

basic ingredient: pointwise gradient estimate for finite-dim. Galerkin approx. L^N of L

$$\|DT_t^N F(u)\|_H \leq \exp\left(\frac{\varepsilon\nu}{4(\nu^2 - \varepsilon\|\sqrt{Q}\|^2_{L(H^2,V)})}\|u\|_V^2 + C(\varepsilon, Q, \nu)\, t\right)\|F\|_{Lip(H)}$$

uniformly in N

Rem $B = 0$ result obtained by Barbu, Da Prato (2004)

white noise $Q = id$

recall invariants for $\Pi(u \cdot \nabla)u)$ and B

- energy $\langle \Pi(u \cdot \nabla)u, u \rangle = 0 = \langle Bu, u \rangle$
- enstrophy $\langle \Pi(u \cdot \nabla)u, A_S u \rangle = 0 = \langle Bu, A_S u \rangle$

implies

$$L_{\sigma,\nu} F(z) = \frac{\sigma^2}{2} \mathrm{tr}\,(D^2 F)(u) + \langle \nu A_S u + Bu - \Pi(u \cdot \nabla)u, DF(u) \rangle$$

has (inf.) invariant measure

$$\mu_{\sigma,\nu} = N(0, \frac{\sigma^2}{\nu}(-A_S)^{-1})$$

results - $Q = id,\ B = 0$

- **Albeverio, Cruzeiro (1990)** - existence of a stationary martingale solution of (7) on $C(\mathbb{R}_+, H^\alpha)$, $\alpha < -\frac{1}{2}$
- **Da Prato, Debussche (2001)** - existence and uniqueness of strong solution of (7) for $\mu_{\sigma,\nu}$-a.e. initial condition
- **Albeverio, Ferrario (2001)** - Uniqueness of self-adjoint dominator

$$L_{\sigma,\nu} F = L^{OU}_{\sigma,\nu} F + \underbrace{\Lambda F}$$
$$\le \frac{1}{2} \sum_k k^{2\varepsilon} \nabla_k^* \nabla_k F + \frac{1}{2} \sum_k k^{-2\varepsilon} \langle u \cdot \nabla u, e_k \rangle^2 F$$

$\le$ in the quadratic form sense

- **S. (JFA 2003)** - L^1-uniqueness of regularized version of $L_{\sigma,\nu}$

$$\partial_t u - \gamma \Delta u + p_\varepsilon \left(p_\varepsilon u \cdot \nabla \right) p_\varepsilon u + \nabla p = \sigma \dot{W}$$

where $p_\varepsilon = e^{\varepsilon \Delta}$, $\varepsilon > 0$ (also for $\nu = \sigma = 0$)

L^1-Uniqueness in the case $B = 0$ - White noise

Theorem (S. IDAQP 2007) Let $\alpha \in (-1, 0)$. Then there exists $C(\alpha)$ such that:

- If $\nu^3 > \sigma^2 C(\alpha)$ for some $\alpha \in (-1, 0)$ then the closure $\bar{L}_{\sigma,\nu}$ generates a Markovian C_0-semigroup on $L^1(\mu_{\sigma,\nu})$.
- The associated resolvent $\bar{R}_{\sigma,\nu,\lambda} = \left(\lambda - \bar{L}_{\sigma,\nu}\right)^{-1}$, $\lambda > 0$, satisfies

$$\int |D\bar{R}_{\sigma,\nu,\lambda} F(u)|^2_{H^{2+\alpha}}\, \mu(du) \le \frac{\nu^2}{\lambda\left(\nu^3 - \sigma^2 C(\alpha)\right)} \int |DF(u)|^2_{H^{1+\alpha}}\, \mu(du)$$

  in part. $\bar{R}_{\sigma,\nu,\lambda}$ operates as a bounded operator

$$\bar{R}_{\sigma,\nu,\lambda} : H^{1,2}_{H^{1+\alpha}}(\mu_{\sigma,\nu}) \to H^{1,2}_{H^{2+\alpha}}(\mu_{\sigma,\nu})$$

  for $\alpha \in (-1, 0)$.

Consequences

- existence of diffusions generated by $\bar{L}_{\sigma,\nu}$ using Dirichlet form techniques
- uniqueness of ass. martingale problem

L^1-Uniqueness in the case $B = 0$ - White noise

Theorem (S. IDAQP 2007) Let $\alpha \in (-1, 0)$. Then there exists $C(\alpha)$ such that:

- If $\nu^3 > \sigma^2 C(\alpha)$ for some $\alpha \in (-1, 0)$ then the closure $\bar{L}_{\sigma,\nu}$ generates a Markovian C_0-semigroup on $L^1(\mu_{\sigma,\nu})$.
- The associated resolvent $\bar{R}_{\sigma,\nu,\lambda} = \left(\lambda - \bar{L}_{\sigma,\nu}\right)^{-1}$, $\lambda > 0$, satisfies

$$\int |D\bar{R}_{\sigma,\nu,\lambda} F(u)|^2_{H^{2+\alpha}}\, \mu(du) \le \frac{\nu^2}{\lambda\left(\nu^3 - \sigma^2 C(\alpha)\right)} \int |DF(u)|^2_{H^{1+\alpha}}\, \mu(du)\,.$$

$B \neq 0$ Sauer (TU Darmstadt): for ν^3/σ^2 sufficiently large

- the same integr. gradient estimates hold
- $(L_{\sigma,\nu}, \mathcal{F}C_b^2)$ is L^1-unique

NB

- invariant measure is independent of angular velocity ω
- can therefore study the limit $|\omega| \to \infty$ in $L^1(\mu_{\sigma,\nu})$

rough noise $Q = (-A_S)^{-\frac{1}{2}-\varepsilon}$

ex. of (stationary) weak martingale solution (should be) similar to the case without Coriolis forcing term (not done)

ex. proof for the stoch. Navier-Stokes without Coriolis forcing term by Flandoli, Gatarek [PTRF, 1995] suggests

$$\int \log\left(1 + \|u\|^2\right)\,\mu(du) < \infty$$

for an invariant distribution

can be improved to

$$\Longrightarrow \qquad \int \left(1 + \|u\|^2_{D((-A_S)^\gamma)}\right) \|u\|_H^{2m}\,\mu(du) < \infty \qquad \forall m \geq 0\,, \gamma < \frac{1}{4} + \varepsilon$$

cf. **Es-Sarhir, S., JFA 2010** Improved moment estimates for inv. measures of semilinear diffusions in Hilbert spaces and applications

improved moment estimates

improved moment estimates for spde of the type

$$dX_t = (AX_t + B(X_t))\, dt + \sqrt{Q}\, dW_t \text{ on } H$$

A self-adjoint, of negative type, compact resolvent satisfying

$$\langle Ay + B(y+w), y\rangle \leq -\alpha_1 \|y\|^2_{V_{\frac{1}{2}}} + \alpha_2 \|w\|^s_{V_{\gamma_0}} + \alpha_3 \|w\|^s_{V_{\gamma_0}} \|y\|^2_{V_{\gamma_1}} + \alpha_4$$

Rem $\alpha_3 = 0$: exponential moments (cf. Es Sarhir, S., JEE 2008)

$\alpha_3 > 0$ - main ingredient

Prop (Da Prato, Debussche, ...) Let $\delta \in]0, \frac{1}{2}[$ and $\gamma \in \mathbb{R}$. Then

$$\sup_{0 \le t \le T} \|W_{A-\lambda}(t)\|_{V_\gamma}^2 \le C_\delta^2 \sum_{k=1}^{\infty} \frac{\lambda_k^{2\gamma}}{(\lambda + \lambda_k)^{2\delta}} M_k(\delta, T)^2$$

where

- $W_{A-\lambda}(t) = \int_0^t e^{(t-s)A} \sqrt{Q}\, dW_s$
- (λ_k) - eigenvalues of $-A$
- $$M_k(\delta, T) := \sup_{0 \le s < t \le T} \frac{|\beta_k(t) - \beta_k(s)|}{|t-s|^\delta}$$
  ind. r.v., having finite moments of any order. Here,
  $$\beta_k(t) := \langle W(t), e_k \rangle \qquad (e_k)_k \text{ eigenfunctions of } A$$
  ind. 1-dim Brownian motions

in part.: if for some $\varepsilon > 0$

$$\sum_{k=1}^{\infty} \lambda_k^{-2(\delta-\gamma-\varepsilon)} < \infty$$

then

$$\sup_{0\le t\le T} \|W_{A-\lambda}(t)\|_{V_\gamma}^2 \le \lambda^{-2\varepsilon} M_{\delta,\gamma,\varepsilon}$$

where $M_{\delta,\gamma,\varepsilon}$ ind. of λ, finite moments

Rem work in progress
similar control on the decay in λ for $\int_0^t e^{(t-s)(A-\lambda)} C\, dW(s)$ und general assumptions on A, C