Example 5.12: Consider the following bilinear stochastic partial differential equation

$$\partial_t u(t,x) = \frac{1}{2}\partial_{xx}^2 u(t,x) + \theta\,\partial_x u(t,x)\,\partial_t w(t) \quad ; \quad u(0,x) = u_0(x).$$

In this case, the coercivity assumption (V.1) is satisfied iff $|\theta| < 1$, since

$$2\langle A(u),u\rangle + \|B(u)\|^2_{L_2(\mathcal{U},H)} = \int \partial_{xx}^2 u\, u\,dx + \theta^2 \int (\partial_x u)^2 dx$$
$$= (\theta^2 - 1)\int (\partial_x u)^2 dx .$$

In particular, the solution $u(t,x)$ will be in $V = H^1(\mathbb{R}^d)$ for all $t>0$, a regularizing effect of parabolic type.

In the critical case $\theta = 1$ one explicit solution is given by

$$u(t,x) = u_0(x + W_t)$$

(and $u(t,x) = u_0(x - W_t)$ in the case $\theta = -1$). Indeed, Itô's formula implies that

$$du(t,x) = \partial_x u(t,x)\,dW_t + \frac{1}{2}\partial_{xx} u(t,x)\,dt.$$

It is clear that $u(t,x) = u_0(x + W_t)$ has the same (spatial) regularity as u_0, so the equation has no regularizing effect and is of hyperbolic type.

In the case $|\theta| > 1$ the sign in front of the term $\int (\partial_x u)^2 dx$ becomes positive, so that the equation essentially becomes a backward equation and cannot be well-posed anymore.

Finally, let us discuss the second method of solving spde's in a variational context: the compactness method:

We keep the assumptions (V.1) and (V.3) and add the following assumptions:

(V.5) $\exists M, \delta > 0$ such that $\forall u \in V$

$$\|B(u)\circ\sqrt{Q}\|_{L_2(U,H)} \leq M\left(1+\|u\|_V^{1-\delta}\right) \qquad \text{(sublinear growth)}$$

(V.6) $V \hookrightarrow H$ is compact

(V.7) $\begin{cases} u \longmapsto A(u) : V_{weak} \cap H \longrightarrow V'_{weak} \text{ is continuous} \\ u \longmapsto B(u) : V_{weak} \cap H \longrightarrow L_2(U,H) \text{ is continuous} \end{cases}$

We now introduce the following concept of a martingale solution of a stochastic partial differential equation:

<u>Definition 5.13:</u> A probability measure P on $(\Omega, \mathcal{F})$ is said to be a solution to the martingale problem associated with the spde (5.6) if

(i) $P(u(0)=u_0) = 1$

(ii) the process

$$M_t = u(t) - u(0) - \int_0^t A(u(s))\,ds$$

is a continuous H-valued P-martingale with associated increasing process

$$\langle M\rangle_t = \int_0^t B(u(s))QB^*(u(s))\,ds.$$

The condition (ii) in the above definition is equivalent with the following: let (e_n) be an orthonormal basis of H with $e_n \in V$ $\forall n$. Then

(ii') $\forall i \geq 1$, $\varphi \in C_b^2(\mathbb{R})$, $0 \leq s \leq t$, Φ_s continuous bounded and $\mathcal{F}_s$-measurable mapping

$$E\left((M_t^{i,\varphi} - M_s^{i,\varphi})\Phi_s\right) = 0, \text{ where}$$

$$M_t^{i,\varphi} := \varphi(\langle u(t), e_i\rangle) - \varphi(\langle u_0, e_i\rangle) - \int_0^t \varphi'(\langle u(s), e_i\rangle)\langle A(u(s)), e_i\rangle\,ds$$
$$+ \frac{1}{2}\int_0^t \varphi''(\langle u(s), e_i\rangle)\langle (BQB^*)(u(s))e_i, e_i\rangle\,ds.$$

<u>Theorem 5.14.</u> Under the assumptions (V.1), (V.3), (V.5), (V.6) and (V.7) there exists a solution P to the martingale problem.

<u>Proof:</u> We consider the same Galerkin approximation as in the proof of Theorem 5.6. Again (See p. 5.15 (5.7))

$V_n = \mathrm{span}\{e_1, \dots, e_n\}$ for some ONB (e_n) of H, contained in V.

π_n := orthog projection in H onto V_n

Then: for each $n \geq 1$, there exists a probability measure $\mathbb{P}_n$ on $(\Omega, \mathcal{F})$ such that

$(0)_n$ $\operatorname{supp}(\mathbb{P}_n) \subseteq C(0,T,V_n)$

$(i)_n$ $\mathbb{P}_n(u(0) = \pi_n u_0) = 1$

$(ii)_n$ $\forall\, \varphi \in C_b^2(\mathbb{R})$, Φ_s cont., bdd., $\mathcal{F}_s$-measurable

$$\mathbb{E}_n\left((M_t^{i,\varphi} - M_s^{i,\varphi})\,\Phi_s\right) = 0$$

$\mathbb{P}_n$ is obtained, solving the finite dimensional SDE (5.7).

Suppose now that the following lemma holds:

<u>Lemma 5.15</u> The sequence of probability measures $(\mathbb{P}_n)_{n\geq 1}$ is tight on Ω.

We can now extract a subsequence - again denoted with $\mathbb{P}_n$ - such that $\mathbb{P}_n \Rightarrow \mathbb{P}$ weakly ($C(0,T;V_n) \subseteq C(0,T;V)$ is a metrizable space).

$\mathbb{P}$ satisfies (i) and the mapping

$$\omega \longmapsto \left(M_t^{i,\varphi}(\omega) - M_s^{i,\varphi}(\omega)\right)\Phi_s(\omega) \text{ is continuous}$$
$$\Omega \longmapsto \mathbb{R}.$$

The coercivity assumption (V.1) implies (Lemma 5.10)

$$\sup_n \mathbb{E}_n\left(\sup_{0\leq t\leq T} \|u(t)\|^2 + \int_0^T \|u(t)\|_V^2\,dt\right) < \infty.$$

In particular, for some $p > 1$ (depending on δ in (V.5)) we get

$$\sup_n \mathbb{E}_n\left(|M_t^{i,\varphi} - M_s^{i,\varphi}|^p\right) < \infty.$$

Indeed:

$$M_t^{i,\varphi} - M_s^{i,\varphi} = \underbrace{\varphi(\langle u_t, e_i\rangle) - \varphi(\langle u_s, e_i\rangle)}_{\text{bdd}}$$

$$+ \int_s^t \underbrace{\varphi'(\langle u_s, e_i\rangle)}_{\text{bdd.}} \underbrace{\langle A(u_s), e_i\rangle}_{\leq M(1+\|u\|_V)\|e_i\|_V \quad (V.3)} ds$$

$$+ \frac{1}{2}\int_s^t \underbrace{\varphi''(\langle u_s, e_i\rangle)}_{\text{bdd.}} \underbrace{\langle (BQB^*)(u_s)e_i, e_i\rangle}_{\substack{\leq \|B(u_s)\sqrt{Q}\|^2_{L_2(U,H)} \\ \leq M(1+\|u_s\|_V^{2(1-\delta)}) \quad (V.5)}} ds$$

So that

$$|M_t^{i,\varphi} - M_s^{i,\varphi}| \leq M_{\delta,t,\varphi}\left(1 + \left(\int_0^T \|u_s\|_V^2 \, ds\right)^{1-\delta}\right)$$

and we can choose $p = \frac{1}{1-\delta}$.

It follows that

$$\lim_{n\to\infty} \mathbb{E}_n\left((M_t^{i,\varphi} - M_s^{i,\varphi})\Phi_s\right) = \mathbb{E}\left((M_t^{i,\varphi} - M_s^{i,\varphi})\Phi_s\right)$$

which implies (ii).

Indeed: For any $K > 0$, $\Psi_K := (-K)\vee(M_t^{i,\varphi} - M_s^{i,\varphi})\wedge K$ is bounded (and continuous), so that

$$\lim_{n\to\infty} \mathbb{E}_n(\Psi_K \Phi_s) = \mathbb{E}(\Psi_K \Phi_s)$$

Moreover,

$$|\mathbb{E}_n((\Psi_K - (M_t^{i,\varphi} - M_s^{i,\varphi}))\Phi_s|$$
$$\leq \mathbb{E}_n\left(|M_t^{i,\varphi} - M_s^{i,\varphi}| 1_{\{|M_t^{i,\varphi} - M_s^{i,\varphi}| \geq K\}}\right)\|\Phi_s\|_\infty$$

$$\leq \frac{1}{k^p} \mathbb{E}_n\left(|M_t^{i,e} - M_s^{i,e}|^p\right) \|\Phi_s\|_\infty$$

$$\leq \frac{1}{k^p} \underbrace{\sup_n \mathbb{E}_n\left(|M_t^{i,e} - M_s^{i,e}|^p\right)}_{<\infty} \|\Phi_s\|_\infty \xrightarrow[k\uparrow\infty]{} 0$$

uniformly in n.

Consequently, $\forall k$

$$\lim_{n\to\infty} \left| \mathbb{E}_n\left((M_t^{i,e} - M_s^{i,e})\Phi_s\right) - \mathbb{E}\left((M_t^{i,e} - M_s^{i,e})\Phi_s\right)\right|$$

$$\leq \overline{\lim_{n\to\infty}} \left\{ \mathbb{E}_n\left(|(M_t^{i,e} - M_s^{i,e}) - \varphi_k|\right) + \mathbb{E}\left(|(M_t^{i,e} - M_s^{i,e}) - \varphi_k|\right)\right\} \|\Phi_s\|_\infty$$

$$\xrightarrow[k\uparrow\infty]{} 0 \quad \blacksquare$$

It remains to give a (sketch) of the proof of <u>lemma 5.15:</u>

Let

τ_1 - weak topology on $L^2(0,T;V)$

τ_2 - uniform topology on $C(0,T;V')$

τ_3 - strong topology on $L^2(0,T;H)$.

We will show that $(\mathbb{P}_n)$ is tight w.r.t. τ_i, $i=1,2,3$. To this end choose

$$K = \left\{ u;\ \sup_{0\leq t\leq T} \|u_t\|_H \leq l,\ \int_0^T \|u_s\|_V^2\, ds \leq k \right\}.$$

The a priori moment estimates on the Galerkin approximations imply that $\mathbb{P}_n(K^c)$ can be made arbitrarily small for large l, k – uniformly in n.

1. τ_1 - tightness: K is relatively compact w.r.t. the weak topology on $L^2(0,T,V)$ (since $L^2(0,T,V)$ is a Hilbert space).

2. τ_2 - tightness: We need to show that for all $h \in V$, $\|h\|_V = 1$, the set of functions

$$\{ t \mapsto \langle u(t), h \rangle , u \in K \}$$

is a compact subset of $C(0,T)$. This follows from Arzela-Ascoli, since:

(a) $$\sup_{u \in K} |\langle u(t), h \rangle| \le l$$

(b) $$\sup_{\substack{u \in K \\ \|h\|_V \le 1}} \underbrace{|\langle u(t) - u(s), h \rangle|}_{} \to 0 \quad , \text{as } |t-s| \to 0$$

$$= \left| \int_s^t \langle A(u_r), h \rangle dr + \int_s^t \langle h, B(u_r) dW_r \rangle \right|$$

$$\le M \left(1 + \int_s^t \|u_r\|_V \, dr \right) + \left| \int_s^t \langle h, B(u_r) dW_r \rangle \right|.$$

In the proof of (b) we need to control the modulus of continuity of the stochastic integrals

$$\int_s^t \langle h, B(u_r) dW_r \rangle \quad , h \in V,$$

uniformly on K. This is beyond the scope of the lecture, but at least in the mean, Burkholder-Davis-Gundy implies that

$$\mathbb{E}_n\left(\left|\int_s^t \langle h, B(u_r)\,dW_r\rangle\right| \right)$$

$$\leq \mathbb{E}_n\left(\sup_{t\in[0,T]} \|u_t\|_H \left(\int_s^t \|B(u_n(r))\circ\sqrt{Q}\|^2_{L_2(U,H)}\,dr\right)^{\frac{1}{2}}\right)$$

$$\leq \mathbb{E}_n\left(\sup_{t\in[0,T]} \|u_t\|_H^2\right)^{\frac{1}{2}} \mathbb{E}_n\Bigg(\underbrace{\int_s^t \|B(u_n(r))\circ\sqrt{Q}\|^2_{L_2(U,H)}\,dr}_{\begin{array}{l}\leq M\left(|t-s| + \int_s^t \|u_n(r)\|_V^{2(1-\delta)}\,dr\right) \\ \leq M\left(|t-s| + |t-s|^{\delta}\underbrace{\left(\int_s^t \|u_n(r)\|_V^2\,dr\right)}_{1-\delta}\right) \\ p=\frac{1}{1-\delta},\ q=\frac{1}{\delta}\end{array}}\Bigg)^{\frac{1}{2}}$$

$$\longrightarrow 0 \quad \text{as } |t-s|\to 0. \qquad \square$$

Finally, again without proof, the τ_3-tightness follows from the τ_i-tightness for $i=1,2$. $\blacksquare$

Ergänzung:

Lemma 5.15a: Given $\{u_n\}$ bdd in $L^2(0,T;V)\cap L^\infty(0,T;H)$ and equicontinuous as V'-valued functions and such that $u_n(0)\to u_0$ strongly in H, one can extract a subsequence converging strongly in $L^2(0,T;H)$.

Proof: We use the following fact:

$\forall\, \varepsilon>0\ \exists C_\varepsilon$ such that

$$\|u\|_H \le \varepsilon \|u\|_V + C_\varepsilon \|u\|_{V'} \quad \forall u\in V.$$

(Indirect: otherwise we could find a $\varepsilon>0$ and a sequence (u_n) such that $\|u_n\|_H \ge \varepsilon\|u_n\|_V + n\|u_n\|_{V'}$.
But then, using $\tilde u_n := \frac{u_n}{\|u_n\|_H}$, and

$$1 = \|\tilde u_n\|_H \ge \varepsilon\|\tilde u_n\|_V + n\|\tilde u_n\|_{V'},$$

we can extract a weakly convergent subsequence $\tilde u_n \rightharpoonup \tilde u\in V$, hence strongly convergent to $\tilde u$ in H. Since

$$\|\tilde u\|_{V'} = \lim_{n\to\infty}\|\tilde u_n\|_{V'} \le \frac{1}{n}\longrightarrow 0$$

we conclude that $\|\tilde u\|_H = 0$. ↯)

We know that there exists a subsequence u_n converging to some $u\in C([0,T];V')$, hence also in $L^2(0,T;V')$. Obviously, $u\in L^2(0,T;V)$, hence for each $\varepsilon>0$ $\exists C(\varepsilon)$ such that

$$\int_0^T \|u_n-u\|_H^2(t)\,dt \le \varepsilon\int_0^T \|u_n-u\|_V^2(t)\,dt + C(\varepsilon)\underbrace{\int_0^T \|u_n-u\|_{V'}^2(t)\,dt}_{\xrightarrow[n\to\infty]{} 0}$$

Thus $\overline{\lim_{n\to\infty}}\int_0^T \|u_n-u\|_H^2\,dt \le \varepsilon$ ∎

Generalizations to Theorem 5.6

We now summarize the generalizations to the main Existence/Uniqueness theorem given in the monograph

Prévôt, Röckner: A concise course on Stochastic Partial Differential Equations, LNM 1905, Springer.

Again, we consider the Gelfand triple $V \hookrightarrow H \cong H' \hookrightarrow V'$ only for separable (real) Hilbert spaces and formulate conditions on existence and uniqueness for solutions of the stochastic evolution equation

$$du(t) = A(t, u(t))\,dt + B(t, u(t))\,dW_t$$

with (W_t) a Q-Wiener process on some Hilbert space $\mathcal{U}$, for

$$A = A(t, u, \omega) : [0,T] \times V \times \Omega \longrightarrow V'$$
$$B = B(t, u, \omega) : [0,T] \times V \times \Omega \longrightarrow L(\mathcal{U}, H)$$

progressively measurable.
We will suppress the dependence on ω and simply write $A(t,u)$, $B(t,u)$. Then the analogous conditions (V.1) – (V.4) now read as follows:

(V.1) $\exists \alpha > 0$, $p \in]1, \infty[$, $\lambda \in \mathbb{R}$ and some adapted process $f \in L^1([0,T] \times \Omega, dt \otimes P)$ such that

$$2 \langle A(t,u), u \rangle + \| B(t,u) \circ \sqrt{Q} \|^2_{L_2(H,H)}$$

$$\leq -\alpha \|u\|_V^p + \lambda \|u\|_H^2 + f(t) \quad \forall u \in V.$$

(V.2) $\exists \lambda \in \mathbb{R}$ such that

$$2 \langle A(t,u_1) - A(t,u_2), u_1 - u_2 \rangle +$$

$$+ \| (B(t,u_1) - B(t,u_2)) \circ \sqrt{Q} \|^2_{L_2(H,H)} \leq \lambda \|u_1 - u_2\|_H^2$$

$$\forall u_1, u_2 \in V$$

(V.3) $\exists M > 0$ and some adapted process $g \in L^{\frac{p}{p-1}}([0,T] \times \Omega, dt \otimes P)$ such that

$$\| A(t,u) \|_{V'} \leq g(t) + M \|u\|_V^{p-1}$$

where p is as in (V.1).

(V.4) $\forall u, v, w \in V, \forall \omega \in \Omega$ the mapping

$$\lambda \longmapsto \langle A(t, u + \lambda v, \omega), w \rangle \quad \text{is continuous.}$$

We can then formulate the following Theorem

Theorem 5.6': Under the assumptions (V.1)–(V.4) there exists for all $u_0 \in H$ a unique $\underset{H\text{-valued continuous}}{\text{}}$ adapted process $u(t)$, $t \in [0,T]$, such that $u \in L^p([0,T]\times\Omega; V) \cap L^2([0,T]\times\Omega; H)$ satisfying

$$\langle u(t), v\rangle = \langle u_0, v\rangle + \int_0^t \langle A(s,u(s)), v\rangle\, ds + \int_0^t \langle v, B(s,u(s))\,dW(s)\rangle .$$

The proof is similar to the original proof. It requires the following generalization of Itô's formula:

Lemma 5.7': Let $u_0 \in H$, u (resp. v) be adapted processes with $u \in L^p([0,T]\times\Omega; V)$ (resp. $v \in L^{\frac{p}{p-1}}([0,T]\times\Omega; V')$), M be continuous local martingale with $E(\langle M\rangle_T) < \infty$ (hence in fact L^2-bounded) such that

$$u(t) = u_0 + \int_0^t v(s)\,ds + M_t .$$

Then:

(i) $u \in C([0,T]; H)$ a.s. and $E\left(\sup_{t\in[0,T]} \|u(t)\|_H^2\right) < \infty$

(ii) $$\|u(t)\|_H^2 = \|u_0\|_H^2 + 2\int_0^t \langle v(s), u(s)\rangle\, ds + 2\int_0^t \langle u(s), dM_s\rangle + \langle M\rangle_t \quad \forall t \in [0,T],\ \text{a.s.}$$

Note that
$$\int_0^t |\langle v(s), u(s)\rangle|\, ds \le \int_0^t \|v(s)\|_{V'} \|u(s)\|_V\, ds \le \left(\int_0^t \|v(s)\|_{V'}^{\frac{p}{p-1}} ds\right)^{\frac{p-1}{p}} \left(\int_0^t \|u(s)\|_V^p\, ds\right)^{\frac{1}{p}}$$

is integrable, hence $\int_0^t \langle v(s), u(s)\rangle\, ds$ well defined. The novelty here is the additional integrability

$E\left(\sup_{t\in[0,T]}\|u(t)\|_H^2\right)<\infty$, this can be proven as follows:

First note that:

$$\|u(t)\|_H^2 = \|u(s)\|_H^2 + 2\int_s^t \langle v(r), u(r)\rangle dr + 2\langle u(s), M(t)-M(s)\rangle$$
$$+\|M(t)-M(s)\|_H^2 - \|u(t)-u(s)-(M(t)-M(s))\|_H^2$$

which implies for $t\geq s$:

$$\|u(t)\|_H^2 \leq 2\|u(s)\|_H^2 + 2\left(\int_0^T \|u(r)\|_V^p dr\right)^{\frac{1}{p}}\left(\int_0^T \|v(r)\|_{V^*}^{\frac{p}{p-1}} dr\right)^{\frac{p-1}{p}}$$
$$+2\|M(t)-M(s)\|_H^2,$$

hence

$$E\left(\sup_{t\in[0,T]}\|u(t)\|_H^2\right)\leq 2\|u_0\|_H^2 + 2E\left(\int_0^T\|u(r)\|_V^p dr\right)^{\frac{1}{p}}E\left(\int_0^T\|v(r)\|_{V^*}^{\frac{p}{p-1}}\right)^{\frac{p-1}{p}}$$
$$+2\underbrace{E\left(\sup_{t\in[0,T]}\|M(t)\|_H^2\right)}_{\leq 8E\left(\|M(T)\|_H^2\right)\ \text{(max. inequality)}}$$

■

Further remarks concerning the proof of <u>Theorem 5.6</u>:

The proof of uniqueness is exactly the same as in Theorem 5.6. Concerning existence we have the following a priori estimate for the Galerkin-approximations:

Lemma 5.10'

$$\sup_n \mathbb{E}\left(\sup_{t\in[0,T]} \|u_n(t)\|_H^2 + \int_0^T \|u_n(s)\|_V^p \, ds \right) < \infty$$

Indeed, in the proof of Lemma 5.10, using ITÔ's formula, we have

$$\langle u_n(t), e_k\rangle^2 = \langle u_0, e_k\rangle^2 + 2\int_0^t \langle A(s,u_n(s)), e_k\rangle \langle u_n(s), e_k\rangle \, ds$$

$$+ 2\int_0^t \langle u_n(s), e_k\rangle \cdot B_k(s,u_n(s))\, dW^n(s)$$

$$+ \sum_{\ell=1}^n \int_0^t B_k(s,u_n(s))(\sqrt{Q} f_\ell)^2 \, ds$$

hence

$$e^{-\lambda t}\|u_n(t)\|_H^2 \le \|u_0\|_H^2 + 2\int_0^t e^{-\lambda s}\left(\langle A(s,u_n(s)), u_n(s)\rangle - \lambda \|u_n(s)\|_H^2\right) ds$$

$$+ 2\int_0^t e^{-\lambda s}\langle u_n(s), B(s,u_n(s))\, dW^n(s)\rangle$$

$$+ \int_0^t e^{-\lambda s}\|B(s,u_n(s))\circ\sqrt{Q}\|_{L_2(U,H)}^2 \, ds$$

$$\le \|u_0\|_H^2 - \alpha\int_0^t e^{-\lambda s}\|u_n(s)\|_V^p \, ds + \int_0^t e^{-\lambda s} f(s)\, ds$$

$$+ 2\int_0^t e^{-\lambda s}\langle u_n(s), B(s,u_n(s))\, dW^n(s)\rangle$$

Taking expectations, we first have that

$$e^{-\lambda t} E\left(\|u_n(t)\|_H^2\right) + \alpha\int_0^t e^{-\lambda s} E\left(\|u_n(s)\|_V^p\right) ds$$

$$\le \|u_0\|_H^2 + \underbrace{E\left(\int_0^t e^{-\lambda s} f(s)\, ds\right)}_{<\infty},$$

thus

$$E\left(\int_0^T \|u_n(s)\|_V^p \, ds\right) \le \frac{e^{\lambda T}}{\alpha}\left(\|u_0\|_H^2 + E\left(\int_0^T |f(s)|\, ds\right)\right) < \infty,$$

uniformly in n.

And then

$$\sup_{s\in[0,t]} e^{-\lambda s}\|u_n\|_H^2 \le \|u_0\|_H^2 + \int_0^t e^{-\lambda s}|f(s)|\,ds + 2\sup_{s\in[0,t]} \int_0^s e^{-\lambda r}\langle u_n(r), B(r,u_n(r))\,dW^n(r)\rangle_H$$

thus – using Burkholder - Gundy inequality –

$$\underbrace{E\left(\sup_{s\in[0,t]} e^{-\lambda s}\|u_n\|_H^2\right)}_{=:\,\Phi(t)} \le \|u_0\|_H^2 + E\left(\int_0^t e^{-\lambda s}|f(s)|\,ds\right)$$

$$+ \frac{1}{2} E\left(\sup_{s\in[0,t]} e^{-\lambda s}\|u_n\|_H^2\right) +$$

$$+ \frac{\mathrm{const}^2}{2}\underbrace{E\left(\int_0^t e^{-\lambda s}\|B(s,u_n(s))\sqrt{Q}\|_{L_2(U,H)}^2\,ds\right)}$$

$$\underset{(V.3)}{\le} \int_0^t e^{-\lambda s}\left(\lambda\|u_n\|_H^2 + f(s) + g(s) + M\|u_n(s)\|_V^p\right)ds$$

hence

$$\Phi(t) \le 2\|u_0\|_H^2 + 2E\left(\int_0^T e^{-\lambda s}\left(2|f(s)| + |g(s)|\right)ds\right)$$

$$+ 2M\,E\left(\int_0^T \|u_n(s)\|_V^p\,ds\right) + \lambda\int_0^t \Phi(s)\,ds$$

which implies the desired boundedness using Gronwall's lemma.

Example 5.16: Consider the stochastic partial differential equations

$$(*)\quad \partial_t u(t,x) = \left[\partial_{xx} u(t,x) + f(t,x,u(t,x))\right] dt + \sigma(t,x,u(t,x)) \frac{\partial w}{\partial t}(t,x)$$

on the unit interval $[0,1]$ with Neumann bdy conditions with the following assumptions:

σ, f are continuous in (t,x,u) satisfying

$$(N.1)\qquad f(t,x,u)u \le -m|u|^p + \lambda u^2 + f_0(t,x) \quad \forall u \in \mathbb{R}$$

for some $f_0 \in L^1([0,T]\times[0,1])$, $\lambda \ge 0$, $p \ge 2$, $m > 0$.

$$(N.3)\qquad |f(t,x,u)| \le g(t,x) + M|u|^{p-1}$$

for some $g \in L^1([0,T]\times[0,1])$, $M \ge 0$

(N.3) implies that the operator $A(t,\cdot): V \to V'$, $V = H^{1,2}(0,1)$,

$$A(t,u)(x) := u_{xx}(x) + f(t,x,u(x))$$

is well-defined using the well-known estimate

$$\|u\|_\infty \le \sqrt{2}\,\|u\|_V,$$

Γ $u \in V \Rightarrow$

$$\left|u(x) - \int_0^1 u(y)\,dy\right| \le \int_0^1 |u(x) - u(y)|\,dy \le \|u_x\|_{L^2}$$

$\llcorner \Rightarrow |u(x)| \le \|u_x\|_{L^2} + \|u\|_{L^2} \le \sqrt{2}\,\|u\|_V.$

since then

$$|\langle A(t,u), v\rangle| \le \left| \int_0^1 u_x v_x\,dx + \int_0^1 f(t,x,u(x))v(x)\,dx \right| \le$$

$$\leq \|u\|_V \|v\|_V + \int_0^1 \left(g(t,x) + \tilde{M}|u(x)|^{p-1}\right)|v|(x)\,dx$$

$$\leq \left(\underbrace{\|u\|_V}_{\leq C_p(1+\|u\|_V^{p-1})} + \int_0^1 g(t,x)\,dx + \tilde{M}\int_0^1 |u(x)|^{p-1}\,dx\right)\|v\|_V \quad (\text{since } p \geq 2)$$

so that $\|A(t,u)\|_{V'} \leq C_p + \int_0^1 g(t,x)\,dx + \tilde{M}\|u\|_V^{p-1}$.

We assume next that:

(N.2) $\quad (f(t,x,u_1) - f(t,x,u_2))(u_1-u_2) \leq \lambda_0 (u_1-u_2)^2,$

$\quad (\sigma(t,x,u_1) - \sigma(t,x,u_2))^2 \leq \lambda_0 (u_1-u_2)^2$

and $\quad \sigma^2(t,x,u) \leq \lambda_0 u^2 \qquad \forall\, u, u_1, u_2 \in \mathbb{R}$

for some $\lambda_0 \in \mathbb{R}$.

Suppose that (W_t) is a Q-Wiener process on H.

Then

$$\|(\sigma(t,x,u_1) - \sigma(t,x,u_2)) \circ \sqrt{Q}\|^2_{L_2(H,H)} =$$

$$= \sum_k \|(\sigma(t,x,u_1) - \sigma(t,x,u_2))\sqrt{Q}e_k\|_H^2$$

$$\leq \sum_k \lambda_0 \int (u_1-u_2)^2(x)(\sqrt{Q}e_k)^2\,dx \leq \lambda_0 \|\sqrt{Q}\|^2_{L_2(H,H)} \|u_1-u_2\|_V^2.$$

Hence, if $\lambda_0 \|\sqrt{Q}\|^2_{L_2(U,H)} \le 1$, it follows that

$$2\langle A(t,u_1)-A(t,u_2),\, u_1-u_2\rangle + \|(\sigma(t,u_1)-\sigma(t,u_2))\circ\sqrt{Q}\|^2_{L_2(U,H)}$$

$$\le -2\int (u_1-u_2)_x^2\, dx + 2\lambda_0 \|u_1-u_2\|^2_{L^2} + \lambda_0 \|\sqrt{Q}\|^2_{L_2(U,H)} \|u_1-u_2\|_V^2$$

$$\le \left(\lambda_0\|\sqrt{Q}\|^2_{L_2(U,H)} - 2\right)\|u_1-u_2\|_V^2 + 2(\lambda_0+1)\|u_1-u_2\|^2_H$$

hence (V.2) satisfied with $\lambda = 2(\lambda_0+1)$.

Similar.

$$2\langle A(t,u),u\rangle + \|\sigma(t,u)\circ\sqrt{Q}\|^2_{L_2(U,H)} \le$$

$$\le -2\|u_x\|^2_{L^2} + 2\lambda_0\|\sqrt{Q}\|^2_{L_2(U,H)}\|u\|_V^2 - m\int|u|^p\, dx + \lambda\int |u|^2\, dx + \int f_0(t,x)\, dx$$

$$\le 2\left(\lambda_0\|\sqrt{Q}\|^2_{L_2(U,H)} - 1\right)\|u\|_V^2 - m\int|u(x)|^p\, dx + (\lambda+2)\int u^2\, dx + \int f_0(t,x)\, dx,$$

So that Theorem 5.6' now implies existence/uniqueness of the strong solution of (*) ■