
Phase Retrieval via Polarization in Dynamical Sampling

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Abstract. In this paper we consider the nonlinear inverse problem of phase retrieval in the context of dynamical sampling. Where phase retrieval deals with the recovery of signals & images from phaseless measurements, dynamical sampling was introduced by Aldroubi et al in 2015 as a tool to recover diffusion fields from spatiotemporal samples. Considering finite-dimensional signals evolving in time under the action of a known matrix, our aim is to recover the signal up to global phase in a stable way from the absolute value of certain space-time measurements. First, we state necessary conditions for the dynamical system of sampling vectors to make the recovery of the unknown signal possible. The conditions deal with the spectrum of the given matrix and the initial sampling vector. Then, assuming that we have access to a specific set of further measurements related to aligned sampling vectors, we provide a feasible procedure to recover almost every signal up to global phase using polarization techniques. Moreover, we show that by adding extra conditions like full spark, the recovery of all signals is possible without exceptions.

Keywords. Phase retrieval, Dynamical frames, Vandermonde matrix, Polarization identity, Dynamical sampling

1 INTRODUCTION

Phase retrieval was introduced in [40] as a problem of reconstructing a signal from its Fourier magnitude and has become increasingly popular in image and signal processing due to its applications in crystallography [30, 32, 39], astronomy [19, 27], and laser optics [43, 44]. In all these applications, phase retrieval occurs as ill-posed inverse problem, where the tremendous ambiguousness is the most critical point. For the classical problem, the non-uniqueness has been well studied, and there are several approaches to surmount this issue by enforcing a priori assumptions or exploit additional measurements, see for instance [2, 16–19, 29, 33, 34, 45, 46] and references therein. Moreover, phase retrieval also occurs in the more abstract setting of frames, where the unknown image has to be recovered from the magnitudes of its frame coefficients. For generic and specific frames, this problem has been studied in [10–13, 21, 22]. The purpose of the

current paper is to consider phase retrieval in the setting of dynamical sampling [4–6, 9], which originate back to sampling and recovering diffusion fields from spatiotemporal measurements [35, 42].

In dynamical sampling, we consider an unknown vector $\mathbf{x} \in \mathbb{C}^d$ that evolves under the action of a matrix $\mathbf{A} \in \mathbb{C}^{d \times d}$ meaning that at time $\ell \in \mathbb{N}$ the signal becomes $\mathbf{x}_\ell = (\mathbf{A}^*)^\ell \mathbf{x}$. Our aim is to recover \mathbf{x} up to global phase from phaseless measurements. More precisely, we want to recover \mathbf{x} from

$$|\langle (\mathbf{A}^*)^\ell \mathbf{x}, \boldsymbol{\phi} \rangle| = |\langle \mathbf{x}, \mathbf{A}^\ell \boldsymbol{\phi} \rangle|, \quad (1)$$

where $\ell = 0, \dots, L-1$ with $L \geq d$, and where $\boldsymbol{\phi} \in \mathbb{C}^d$ is some sampling vector. Phase retrieval in dynamical sampling has been already considered for real Hilbert spaces in [7, 8], where the authors provided conditions to ensure that $\{\mathbf{A}^\ell \boldsymbol{\phi}\}_{\ell=0}^{L-1}$ has the complementary property meaning that each subset or its complement spans the entire space. The results have then been generalized to several sampling vectors. The complementary property is here equivalent to the uniqueness (up to global phase) of phase retrieval from (1). However, their techniques cannot be immediately generalized to the complex setting since here the complementary property is not sufficient [13, 14].

To insure that we can do phase retrieval in \mathbb{C}^d , we will assume that $\{\mathbf{A}^\ell \boldsymbol{\phi}\}_{\ell=0}^{L-1}$ is a frame, and we will align $\boldsymbol{\phi}$ with specifically chosen additional sampling vectors to exploit polarization techniques. This idea is inspired by interferometry used in [10, 15]. Using the extra information, we first recover the frame coefficients $\langle \mathbf{x}, \mathbf{A}^\ell \boldsymbol{\phi} \rangle$ up to global phase and then recover \mathbf{x} in a stable way via the dual frame of $\{\mathbf{A}^\ell \boldsymbol{\phi}\}_{\ell=0}^{L-1}$.

The paper is organized as follows. In Section 2 we set the stage by providing the necessary background information about polarization identities, frames and Vandermonde matrices. In Section 3 we find conditions on the spectrum of \mathbf{A} and the vector $\boldsymbol{\phi}$ such that the iterated set $\{\mathbf{A}^\ell \boldsymbol{\phi}\}_{\ell=0}^{L-1}$ is a frame. Moreover, in Section 4, we provide conditions under which this frame has full spark. In Section 5 we prove that the aligned sampling vectors allow phase retrieval for almost all $\mathbf{x} \in \mathbb{C}^d$; moreover, if the underlying dynamical frame has full spark, the recovery of all $\mathbf{x} \in \mathbb{C}^d$ is possible.

2 PRELIMINARIES

2.1 POLARIZATION AND RELATIVE PHASES Our main results are based on the following polarization technique, which allow the recovery of the lost phases from certain phaseless information.

THEOREM 2.1 (POLARIZATION, [15]). *Let $\alpha_1, \alpha_2 \in \mathbb{R}$ satisfy $\alpha_1 - \alpha_2 \notin \pi\mathbb{Z}$. Then, for every $z_1, z_2 \in \mathbb{C} \setminus \{0\}$, the product $\bar{z}_1 z_2$ is uniquely determined by*

$$|z_1|, \quad |z_2|, \quad |z_1 + e^{i\alpha_1} z_2|, \quad |z_1 + e^{i\alpha_2} z_2|.$$

Proof. On the basis of the polar decomposition $z_\ell = |z_\ell| e^{i\phi_\ell}$ with $\ell \in \{1, 2\}$, the last two absolute values are equivalent to

$$|z_1 + e^{i\alpha_\ell} z_2|^2 = |z_1|^2 + |z_2|^2 + 2|z_1||z_2|\Re[e^{i(\phi_2 - \phi_1 + \alpha_\ell)}].$$

Since z_1 and z_2 are non-zero, we can thus extract the real parts

$$r_\ell := \Re[e^{i(\phi_2 - \phi_1 + \alpha_\ell)}].$$

Using Euler's formula, we obtain the linear equation system

$$\begin{aligned} r_1 &= \cos(\alpha_1) \cos(\phi_2 - \phi_1) - \sin(\alpha_1) \sin(\phi_2 - \phi_1) \\ r_2 &= \cos(\alpha_2) \cos(\phi_2 - \phi_1) - \sin(\alpha_2) \sin(\phi_2 - \phi_1). \end{aligned}$$

The determinant of the system matrix is here

$$\det \begin{pmatrix} \cos \alpha_1 & -\sin \alpha_1 \\ \cos \alpha_2 & -\sin \alpha_2 \end{pmatrix} = \sin(\alpha_1 - \alpha_2),$$

which is non-zero by assumption; so $\cos(\phi_2 - \phi_1)$ and $\sin(\phi_2 - \phi_1)$ are uniquely determined by the given data. Knowing the relative phase $\phi_2 - \phi_1$, we calculate $\bar{z}_1 z_2$. \square

Remark 2.2 (Real polarization). For every $z_1, z_2 \in \mathbb{R} \setminus \{0\}$, the product $z_1 z_2$ is uniquely determined by $|z_1|$, $|z_2|$, and $|z_1 + \alpha z_2|$ with $\alpha \in \{-1, 1\}$ because $|z_1 + \alpha z_2|^2 = z_1^2 + z_2^2 + 2\alpha z_1 z_2$.

Remark 2.3 (Polarization identities [10, 15]). For certain α_1 and α_2 as in Theorem 2.1, the phase of z_1, z_2 can be computed without solving a linear equation system. More generally, if ζ_K is chosen to be the K th root of unity, then we have

$$\bar{z}_1 z_2 = \frac{1}{K} \sum_{k=0}^{K-1} \zeta_K^k |z_1 + \zeta_K^{-k} z_2|^2.$$

2.2 FRAMES Given a matrix $A \in \mathbb{C}^{d \times d}$ and a vector $\phi \in \mathbb{C}^d$, the set $\{A^\ell \phi\}_{\ell=0}^{L-1}$ is called a *dynamical frame* if it spans \mathbb{C}^d . Dynamical frames were first introduced in [4] in order to recover a signal evolving in time from certain time-space measurements, where also the infinite dimensional problem is addressed. The topic was further developed in [3, 5, 6, 9, 20, 24–26, 41, 42]. An arbitrary vector $x \in \mathbb{C}^d$ can be recovered from the set $\{\langle A^\ell x, \phi \rangle\}_{\ell=0}^{L-1}$ in a stable way if there exists $\alpha, \beta > 0$ such that

$$\alpha \|\mathbf{y}\|^2 \leq \|\{\langle A^\ell \mathbf{y}, \phi \rangle\}_{\ell=0}^{L-1}\|^2 \leq \beta \|\mathbf{y}\|^2, \quad \text{for all } \mathbf{y} \in \mathbb{C}^d,$$

i.e., when the set $\{(A^T)^\ell \phi\}_{\ell=0}^{L-1}$ is a frame for \mathbb{C}^d .

For any dynamical frame $\{A^\ell \phi\}_{\ell=0}^{L-1}$ there exists a set of vectors in the form $\{B^\ell \tilde{\phi}\}_{\ell=0}^{L-1}$ such that every $x \in \mathbb{C}^d$ can be written as

$$x = \sum_{\ell=0}^{L-1} \langle x, A^\ell \phi \rangle B^\ell \tilde{\phi}. \quad (2)$$

Indeed for a given frame $\{A^\ell \phi\}_{\ell=0}^{L-1}$, the frame matrix $T := \sum_{\ell=0}^{L-1} A^\ell \phi (A^\ell \phi)^*$ is symmetric and positive definite and the canonical dual frame $\{T^{-1} A^\ell \phi\}_{\ell=0}^{L-1}$ can be written in the form $\{B^\ell \tilde{\phi}\}_{\ell=0}^{L-1}$ where $B := T^{-1} A T$ and $\tilde{\phi} := T^{-1} \phi$. For more information about frames, we refer to [23].

Example 2.4. Let $d = 2$, and consider the rotation matrix $A = \begin{pmatrix} \cos \theta & -\sin \theta \\ \sin \theta & \cos \theta \end{pmatrix}$ with $\theta \neq k\pi$ for $k \in \mathbb{Z}$. For every nonzero vector $\phi \in \mathbb{C}^2$ and $L \geq 2$ the set $\{A^\ell \phi\}_{\ell=0}^{L-1}$ is a dynamical frame for \mathbb{C}^2 .

2.3 VANDERMONDE MATRICES As we will see in Section 3, the frame property of the set $\{A^\ell \phi\}_{\ell=0}^{L-1}$ is highly related to the Vandermonde matrix generated by the vector λ whose coordinates consists of the eigenvalues of the matrix A . There are different types of Vandermonde matrices in the literature. We will need the following kinds.

THE CLASSICAL VANDERMONDE MATRIX For $\lambda := (\lambda_0, \dots, \lambda_{d-1})^T \in \mathbb{C}^d$, the Vandermonde matrix $V_\lambda \in \mathbb{C}^{d \times L}$ generated by λ is defined as

$$V_\lambda := (\lambda_k^\ell)_{k,\ell=0}^{d-1,L-1}.$$

The determinant of a square Vandermonde matrix $V_\lambda \in \mathbb{C}^{d \times d}$ equals to

$$\det V_\lambda = \prod_{0 \leq k < j \leq d-1} (\lambda_k - \lambda_j).$$

GENERALIZATION OF THE FIRST KIND A generalized Vandermonde matrix of the first kind is a matrix consisting of selective columns of V_λ . More precisely, for a vector $\lambda := (\lambda_0, \dots, \lambda_{d-1})^T \in \mathbb{C}^d$ and $\mathbf{m} := (m_0, \dots, m_{L-1})^T \in \mathbb{N}_0^L$, the Vandermonde matrix $V_{\lambda, \mathbf{m}} \in \mathbb{C}^{d \times L}$ is defined as

$$V_{\lambda, \mathbf{m}} := (\lambda_k^{m_\ell})_{k,\ell=0}^{d-1,L-1},$$

The Vandermonde determinant of the first kind may be factorized by

$$\det V_{\lambda, \mathbf{m}} = \left(\prod_{k>j} (\lambda_k - \lambda_j) \right) S(\lambda). \quad (3)$$

where S is a symmetric polynomial in λ with non-negative, integer coefficients [28]. The occurring polynomials S are better known as Schur functions [36, 37].

GENERALIZATION OF THE SECOND KIND The second kind generalized Vandermonde matrix $\tilde{V}_{\lambda, \mathbf{m}} \in \mathbb{C}^{d \times L}$ is defined as

$$\tilde{V}_{\lambda, \mathbf{m}} := \begin{bmatrix} \mathbf{R}_0 \\ \vdots \\ \mathbf{R}_{M-1} \end{bmatrix} \quad \text{with} \quad \mathbf{R}_j := \left(\binom{\ell}{k} \lambda_j^{\ell-k} \right)_{k, \ell=0}^{m_j-1, L-1},$$

where $M \in \mathbb{N}$, $\lambda \in \mathbb{C}^M$ and $\mathbf{m} \in \mathbb{N}^M$ such that $|\mathbf{m}| := \sum_{j=0}^{M-1} m_j = d$. Clearly if \mathbf{m} is the unite vector, i.e., $\mathbf{m} = (1, \dots, 1)^T \in \mathbb{N}^M$, then $\tilde{V}_{\lambda, \mathbf{m}}$ equals the Vandermonde matrix V_{λ} . The determinant is given by

$$\det(\tilde{V}_{\lambda, \mathbf{m}}) = \prod_{0 \leq k < j \leq M-1} (\lambda_j - \lambda_k)^{m_k m_j}, \quad (4)$$

see [1, 31]. Obviously, a square Vandermonde matrix $\tilde{V}_{\lambda, \mathbf{m}}$ is invertible precisely when λ has distinct elements.

Example 2.5. For $\lambda := (\lambda_0, \lambda_1, \lambda_2)^T$, $\mathbf{m} := (3, 1, 2)^T$ and $L = d = 6$, we have

$$\tilde{V}_{\lambda, \mathbf{m}} := \begin{bmatrix} 1 & \lambda_0 & \lambda_0^2 & \lambda_0^3 & \lambda_0^4 & \lambda_0^5 \\ 0 & 1 & 2\lambda_0 & 3\lambda_0^2 & 4\lambda_0^3 & 5\lambda_0^4 \\ 0 & 0 & 1 & 3\lambda_0 & 6\lambda_0^2 & 10\lambda_0^3 \\ 1 & \lambda_1 & \lambda_1^2 & \lambda_1^3 & \lambda_1^4 & \lambda_1^5 \\ 1 & \lambda_2 & \lambda_2^2 & \lambda_2^3 & \lambda_2^4 & \lambda_2^5 \\ 0 & 1 & 2\lambda_2 & 3\lambda_2^2 & 4\lambda_2^3 & 5\lambda_2^4 \end{bmatrix}.$$

3 DYNAMICAL FRAMES

To recover a signal from (1), we first study conditions on the matrix $\mathbf{A} \in \mathbb{C}^{d \times d}$ and the vector $\boldsymbol{\phi} \in \mathbb{C}^d$ such that $\{\mathbf{A}^\ell \boldsymbol{\phi}\}_{\ell=0}^{L-1}$ is a frame for \mathbb{C}^d . The cornerstone is here the Jordan canonical form of \mathbf{A} . More precisely, every matrix $\mathbf{A} \in \mathbb{C}^{d \times d}$ is similar to a so-called Jordan matrix meaning that there exists an invertible matrix $\mathbf{S} \in \mathbb{C}^{d \times d}$ such that $\mathbf{A} = \mathbf{S}\mathbf{J}\mathbf{S}^{-1}$ and $\mathbf{J} \in \mathbb{C}^{d \times d}$ is a blocked diagonal matrix of the form

$$\mathbf{J} = \text{diag}(\mathbf{J}_0, \dots, \mathbf{J}_{M-1}) \quad \text{with} \quad \mathbf{J}_j = \begin{pmatrix} \lambda_j & 1 & & \\ & \lambda_j & \ddots & \\ & & \ddots & 1 \\ & & & \lambda_j \end{pmatrix} \in \mathbb{C}^{m_j \times m_j},$$

where λ_j is the j th eigenvalue and m_j the corresponding algebraic multiplicity, and where the columns of $\mathbf{S} = [\mathbf{S}_0 | \dots | \mathbf{S}_{M-1}]$ with blocks $\mathbf{S}_j = [\mathbf{s}_{j,0} | \dots | \mathbf{s}_{j,m_j-1}]$ span the generalized eigenspaces of \mathbf{A} . Further, we have $(\mathbf{A} - \lambda_j \mathbf{I})^{k+1} \mathbf{s}_{j,k} = 0$ but $(\mathbf{A} - \lambda_j \mathbf{I})^k \mathbf{s}_{j,k} \neq 0$

for $k = 0, \dots, m_j - 1$. The Jordan chain S_j related to λ_j is generated by \mathbf{s}_{j,m_j-1} via $\mathbf{s}_{j,k} = (\mathbf{A} - \lambda_j \mathbf{I})^{m_j-k-1} \mathbf{s}_{j,m_j-1}$. We say that $\boldsymbol{\phi}$ depends on the Jordan generator or leading generalized eigenvector \mathbf{s}_{j,m_j-1} if $(S^{-1}\boldsymbol{\phi})_{k-1} \neq 0$ where $k = \sum_{i=0}^{j-1} m_i$. For pairwise distinct eigenvalues λ_j as usually assumed in the following, the generators are unique up to scaling. In this case, S is unique up to scaling and permutation of the blocks S_j . Finally we notice that the ℓ th power of a Jordan matrix and the corresponding Jordan blocks are given by

$$J^\ell = \text{diag}(J_0^\ell, \dots, J_{M-1}^\ell) \quad \text{with} \quad J_j^\ell = \left(\binom{\ell}{n-k} \lambda_j^{\ell-n+k} \right)_{k,n=0}^{m_j-1}.$$

The following two theorems are special cases of [4], where the construction of a frame by iterated actions of \mathbf{A} on a finite set of sampling vectors $\{\boldsymbol{\phi}_j\} \subset \mathbb{C}^d$ is studied. In difference to [4], we provide brief, direct proofs based on the Vandermonde determinant for the case that \mathbf{A} acts on a single generator $\boldsymbol{\phi}$.

THEOREM 3.1 (DYNAMICAL BASIS). *Let $\mathbf{A} \in \mathbb{C}^{d \times d}$ be arbitrary. Then $\{\mathbf{A}^\ell \boldsymbol{\phi}\}_{\ell=0}^{d-1}$ is a basis if and only if the eigenvalues of the Jordan blocks of \mathbf{A} are pairwise distinct and $\boldsymbol{\phi}$ depends on all Jordan generators.*

Proof. Assume that \mathbf{A} has the Jordan decomposition $\mathbf{A} = \mathbf{S}\mathbf{J}\mathbf{S}^{-1}$. We represent the vector $\boldsymbol{\phi}$ with respect to the column-wise basis in S according to the size of the Jordan blocks in J . More precisely, we denote by $\boldsymbol{\psi}_j$ the coordinates corresponding to the basis vectors in S_j . The coefficients are thus given by

$$\boldsymbol{\psi} := \begin{pmatrix} \boldsymbol{\psi}_0 \\ \vdots \\ \boldsymbol{\psi}_{M-1} \end{pmatrix} = \mathbf{S}^{-1}\boldsymbol{\phi}.$$

Next, we consider the generated vectors $\boldsymbol{\phi}_\ell := \mathbf{A}^\ell \boldsymbol{\phi}$ with $\ell = 0, \dots, d-1$. On the basis of the Jordan canonical form, they are given by $\boldsymbol{\phi}_\ell = \mathbf{S}\mathbf{J}^\ell \boldsymbol{\psi}$. Considering only the j th Jordan block, we notice

$$J_j^\ell \boldsymbol{\psi}_j = \mathbf{H}(\boldsymbol{\psi}_j) \left(\binom{\ell}{k} \lambda_j^{\ell-k} \right)_{k=0}^{m_j-1}, \quad (5)$$

where

$$\mathbf{H}(\boldsymbol{\psi}_j) = \begin{bmatrix} (\boldsymbol{\psi}_j)_0 & (\boldsymbol{\psi}_j)_1 & \dots & (\boldsymbol{\psi}_j)_{m_j-1} \\ \vdots & \vdots & \ddots & \vdots \\ (\boldsymbol{\psi}_j)_{m_j-2} & (\boldsymbol{\psi}_j)_{m_j-1} & & \\ (\boldsymbol{\psi}_j)_{m_j-1} & 0 & \dots & 0 \end{bmatrix},$$

is an upper-left Hankel matrix in $\mathbb{C}^{m_j \times m_j}$. The vector on the right-hand side of (5) is here the ℓ th column of R_j within the definition of generalized Vandermonde matrix $\tilde{V}_{\lambda, m}$. The matrix of the generated vectors may hence be written as

$$[\phi_0 | \dots | \phi_{d-1}] = S \begin{bmatrix} H(\psi_0) & \dots & \mathbf{0} \\ \vdots & \ddots & \vdots \\ \mathbf{0} & \dots & H(\psi_{m-1}) \end{bmatrix} \tilde{V}_{\lambda, m}.$$

This matrix is invertible if and only if the generalized Vandermonde matrix $\tilde{V}_{\lambda, m}$ is invertible, i.e. if the eigenvalues are pairwise distinct, see (4), and if the Hankel matrices $H(\psi_j)$ are regular, i.e. if the coefficients $(\psi_j)_{m_j-1}$ of the highest-order generalized eigenvectors do not vanish. \square

THEOREM 3.2 (DYNAMICAL FRAME). *Let $L \geq d$, and let $A \in \mathbb{C}^{d \times d}$ be arbitrary. Then $\{A^\ell \phi\}_{\ell=0}^{L-1}$ is a frame if and only if the eigenvalues of the Jordan blocks of A are pairwise distinct and ϕ depends on all Jordan generators.*

Proof. If the vector ϕ is independent of one Jordan generator, then the images $A^\ell \phi$ are also independent of this generator; so $\{A^\ell \phi\}_{\ell=0}^{L-1}$ can not be a frame for \mathbb{C}^d . Now assume that some eigenvalues of A coincide, i.e. the Jordan block to this eigenvalue decompose into several smaller Jordan blocks. Assume that the eigenvalues λ_{j_0} and λ_{j_1} coincide, and that the corresponding Jordan blocks have dimension $m_{j_0} \times m_{j_0}$ and $m_{j_1} \times m_{j_1}$. Using the notation in the proof of Theorem 3.1, the coordinates of ϕ in $E := \text{span}\{\mathbf{s}_{j_0, m_{j_0}-1}, \mathbf{s}_{j_1, m_{j_1}-1}\}$ are $(\psi_{j_0})_{m_{j_0}-1}$ and $(\psi_{j_1})_{m_{j_1}-1}$. Applying A^ℓ to ϕ , we get the coordinates $\lambda_{j_0}^\ell (\psi_{j_0})_{m_{j_0}-1}$ and $\lambda_{j_1}^\ell (\psi_{j_1})_{m_{j_1}-1}$ with $\lambda_{j_0} = \lambda_{j_1}$ regarding the subspace E . Thus the projections $\text{proj}_E(\{A^\ell \phi\}_{\ell=0}^{L-1})$ only span a one-dimensional subspace. As a consequence $\{A^\ell \phi\}_{\ell=0}^{L-1}$ cannot span \mathbb{C}^d , and we cannot obtain a frame. The opposite direction has already been proven with Theorem 3.1. \square

Since generic matrices $A \in \mathbb{C}^{d \times d}$ are diagonalizable with pairwise distinct eigenvalues, for almost all matrices holds the following special case.

COROLLARY 3.3 (DYNAMICAL FRAME). *Let $L \geq d$, and let $A \in \mathbb{C}^{d \times d}$ be diagonalizable. Then $\{A^\ell \phi\}_{\ell=0}^{L-1}$ is a frame if and only if the eigenvalues of A are pairwise distinct and ϕ depends on all eigenvectors.*

Proof. Since the Jordan blocks here reduces to size 1×1 , the matrix of the generated vectors in the proof of Theorem 3.1 simplifies to

$$[\phi | A\phi | \dots | A^{d-1}\phi] = S \text{diag}(\psi) V_\lambda.$$

This matrix is invertible if and only if the classical Vandermonde matrix $V_\lambda \in \mathbb{C}^{d \times d}$ is invertible and none of the coordinates of ψ vanishes. \square

For $\mathbf{a} \in \mathbb{C}^d$, let $\text{circ}(\mathbf{a})$ denote the circulant matrix whose first column is given by the vector \mathbf{a} . All circulant matrices are diagonalizable with respect to the discrete Fourier transform, i.e.,

$$\text{circ}(\mathbf{a}) = \frac{1}{d} \mathbf{F} \text{diag}(\hat{\mathbf{a}}) \mathbf{F}^{-1},$$

where $\hat{\mathbf{a}} = \mathbf{F}\mathbf{a}$ is given via the Fourier matrix $\mathbf{F} = (e^{-\frac{2\pi ijk}{d}})_{j,k=0}^{d-1}$.

COROLLARY 3.4 (REPEATED CONVOLUTION). *Let $L \geq d$, and let $\phi, \mathbf{a} \in \mathbb{C}^d$ be arbitrary. Then the family*

$$\left\{ \underbrace{\mathbf{a} * \cdots * \mathbf{a}}_{\ell \text{ times}} * \phi \right\}_{\ell=0}^{L-1}$$

is a frame for \mathbb{C}^d if and only if the coordinates of $\hat{\phi}$ do not vanish and the coordinates of $\hat{\mathbf{a}}$ are pairwise distinct.

Proof. Note that $\mathbf{a} * \phi = \text{circ}(\mathbf{a})\phi$ and $\mathbf{A} := \text{circ}(\mathbf{a})$ is a diagonalizable matrix that by hypothesis has pairwise distinct eigenvalues $\{\hat{a}_k\}_{k=0}^{d-1}$. The result follows now from Corollary 3.3. \square

4 FULL-SPARK DYNAMICAL FRAMES

A frame $\{f_k\}_{k=0}^{L-1}$ has full spark if every subset embracing d elements spans \mathbb{C}^d . This property makes full-spark frames attractive in phase retrieval and more generally in signal processing [10, 11, 13, 38]. In the following, we study conditions ensuring that frames generated via diagonalizable matrices have full spark. We show that a dynamical frame has full spark precisely when the Vandermonde matrix \mathbf{V}_λ related to the eigenvalues of \mathbf{A} has full spark.

THEOREM 4.1. *Let $\mathbf{A} \in \mathbb{C}^{d \times d}$ be diagonalizable with eigenvalues λ . For every $L \geq d$, the set $\{\mathbf{A}^\ell \phi\}_{\ell=0}^{L-1}$ is a full spark frame if and only if ϕ depends on all eigenvectors and $\mathbf{V}_\lambda \in \mathbb{C}^{d \times L}$ has full spark.*

Proof. Assume that \mathbf{A} has the eigenvalue decomposition $\mathbf{A} = \mathbf{S}\mathbf{J}\mathbf{S}^{-1}$, where \mathbf{J} is a diagonal matrix, and denote the coordinates of ϕ with respect to \mathbf{S} by $\psi := \mathbf{S}^{-1}\phi$. Consider an arbitrary subset $\{\mathbf{A}^{m_\ell} \phi\}_{\ell=0}^{d-1}$ of $\{\mathbf{A}^\ell \phi\}_{\ell=0}^{L-1}$ with $\mathbf{m} = (m_0, \dots, m_{d-1})^T$. Then the matrix

$$[\mathbf{A}^{m_0} \phi | \mathbf{A}^{m_1} \phi | \dots | \mathbf{A}^{m_{d-1}} \phi] = \mathbf{S} \text{diag}(\psi) \mathbf{V}_{\lambda, \mathbf{m}}.$$

is invertible if and only if all elements of ψ are non-zero and if $\mathbf{V}_{\lambda, \mathbf{m}}$ is invertible, which means that \mathbf{V}_λ has full spark. \square

The following result specializes Theorem 4.1 for \mathbf{A} with eigenvalues $\lambda_k = \lambda^k$.

COROLLARY 4.2. *Let $A \in \mathbb{C}^{d \times d}$ be diagonalizable with eigenvalues $\lambda = (\lambda^k)_{k=0}^{d-1}$ with $\lambda^k \neq 1$ for some $\lambda \in \mathbb{C}$. For every $L \geq d$, the set $\{A^\ell \phi\}_{\ell=0}^{L-1}$ is a full spark frame if and only if ϕ depends on all eigenvectors.*

Proof. For the chosen λ , every $d \times d$ sub-matrix of V_λ is an invertible Vandermonde matrix. \square

Example 4.3. Let $L \geq d$ and $\lambda = e^{2\pi i/L}$ be the L th unit root. Consider the matrix $A = \text{diag}(\lambda^0, \dots, \lambda^{d-1})$ and $\phi = \mathbf{1}$. Then the set $\{A^\ell \phi\}_{\ell=0}^{L-1}$ is a frame for \mathbb{C}^d and has full spark by Corollary 3.3 and Corollary 4.2. This frame is called harmonic and is related to a submatrix of the discrete Fourier matrix. In general not every submatrix of the discrete Fourier transform matrix forms a full-spark frame. For more information we refer to [11].

THEOREM 4.4. *Let $L \geq d$, and let $A \in \mathbb{C}^{d \times d}$ be diagonalizable with distinct real and non-negative eigenvalues. Then $\{A^\ell \phi\}_{\ell=0}^{L-1}$ is a full-spark frame if ϕ depends on all eigenspaces.*

Proof. Due to Theorem 4.1, the set $\{A^\ell \phi\}_{\ell=0}^{L-1}$ has full spark if and only if the generalized Vandermonde matrices $V_{\lambda, \mathbf{m}}$ are invertible for every $\mathbf{m} \in \mathbb{N}_0^d$ with distinct coordinates. Since the Schur functions in (3) have only non-negative coefficients, the generalized Vandermonde determinant is here positive for all λ with non-negative, distinct coordinates, which establishes the assertion. \square

5 PHASE RETRIEVAL IN DYNAMICAL SAMPLING

As mentioned in the introduction, the complementary property can be exploited to ensure phases retrieval for real signals [7,8] Since this approach fails in the complex setting, we align ϕ with further sampling vectors allowing polarization. This allow us to recover the frame coefficient $\langle x, A^\ell \phi \rangle$ up to global phase and then using the frame property we can reconstruct x .

THEOREM 5.1. *Let $\{A^\ell \phi\}_{\ell=0}^{L-1}$ be a frame for \mathbb{C}^d , and let $\alpha_1, \alpha_2 \in \mathbb{R}$ be real numbers with $\alpha_1 - \alpha_2 \notin \pi\mathbb{Z}$. Then almost all $x \in \mathbb{C}^d$ can be recovered from*

$$\{|\langle x, A^\ell \phi \rangle|\}_{\ell=0}^{L-1} \cup \{|\langle x, A^\ell (\phi + e^{i\alpha_k} A\phi) \rangle|\}_{\ell=0, k=1}^{L-2, 2}$$

up to global phase.

Proof. We consider the dense set of $x \in \mathbb{C}^d$ for which $\langle x, A^\ell \phi \rangle \neq 0$ for $\ell = 0, \dots, L-1$. Using the polarization in Theorem 2.1, we determine the products

$$\overline{\langle x, A^\ell \phi \rangle} \langle x, A^{\ell+1} \phi \rangle \quad (\ell = 0, \dots, L-2).$$

Considering the phase of the above identity, we calculate the relative phases

$$\arg \langle \mathbf{x}, \mathbf{A}^{\ell+1} \boldsymbol{\phi} \rangle - \arg \langle \mathbf{x}, \mathbf{A}^{\ell} \boldsymbol{\phi} \rangle \pmod{2\pi} \quad (\ell = 0, \dots, L-2).$$

Choosing the phase of $\langle \mathbf{x}, \boldsymbol{\phi} \rangle$ arbitrary, we may thus recover the frame coefficients $\langle \mathbf{x}, \mathbf{A}^{\ell} \boldsymbol{\phi} \rangle$ up to global phase and thus \mathbf{x} . \square

COROLLARY 5.2. *If $\{\mathbf{A}^{\ell} \boldsymbol{\phi}\}_{\ell=0}^{L-1}$ is a frame for \mathbb{R}^d , and $\alpha \in \{-1, 1\}$, then almost every $\mathbf{x} \in \mathbb{R}^d$ can be recovered from*

$$\{|\langle \mathbf{x}, \mathbf{A}^{\ell} \boldsymbol{\phi} \rangle|\}_{\ell=0}^{L-1} \cup \{|\langle \mathbf{x}, \mathbf{A}^{\ell} (\boldsymbol{\phi} + \alpha \mathbf{A} \boldsymbol{\phi}) \rangle|\}_{\ell=0}^{L-2}$$

up to sign.

Although the extended measurement set allows the extraction of relative phases, the proposed procedure may fail in rare cases, where some of the coefficients $|\langle \mathbf{x}, \mathbf{A}^{\ell} \boldsymbol{\phi} \rangle|$ are zero for some ℓ which means that we are not able to recover \mathbf{x} if it lies in the union of finitely many hyperplanes. On the contrary, if the generated frame has full-spark, one do not need all of the coefficients to recover the wanted signal.

THEOREM 5.3. *Let $\{\mathbf{A}^{\ell} \boldsymbol{\phi}\}_{\ell=0}^{L-1}$ be a full-spark frame for \mathbb{C}^d , and let $\alpha_1, \alpha_2 \in \mathbb{R}$ be real numbers with $\alpha_1 - \alpha_2 \notin \pi\mathbb{Z}$. If $L \geq d^2/4 + d/2$, then every $\mathbf{x} \in \mathbb{C}^d$ can be recovered from the samples*

$$\{|\langle \mathbf{x}, \mathbf{A}^{\ell} \boldsymbol{\phi} \rangle|\}_{\ell=0}^{L-1} \cup \{|\langle \mathbf{x}, \mathbf{A}^{\ell} (\boldsymbol{\phi} + e^{i\alpha_k} \mathbf{A} \boldsymbol{\phi}) \rangle|\}_{\ell=0, k=1}^{L-2, 2}$$

up to global phase.

Proof. Since $\{\mathbf{A}^{\ell} \boldsymbol{\phi}\}_{\ell=0}^{L-1}$ is a full-spark frame, we only need to know the phase of d coefficients $\langle \mathbf{x}, \mathbf{A}^{\ell} \boldsymbol{\phi} \rangle$ to recover \mathbf{x} . Obviously, if at least d coefficients are zero, then the unknown signal is zero everywhere. Now assume that $m < d$ measurements $|\langle \mathbf{x}, \mathbf{A}^{\ell} \boldsymbol{\phi} \rangle|$ are zero. As soon as we find $d - m$ consecutive non-zero measurements, we can transfer the relative phases to enough frame elements to recover \mathbf{x} using the extended measurement set. In the worst case, we measure $d - m - 1$ consecutive non-zeros followed by a zero. After this pattern has been repeated m times, the remaining measurements have to be non-zero. If we thus have at least $L \geq (m+1)(d-m)$ measurements, the existence of at least $d - m$ non-zero consecutive measurements is guaranteed. Considering that the maximum over $(m+1)(d-m)$ is attained at $m := (d-1)/2$ for odd d and $m := d/2$ for even d finishes the proof. \square

THEOREM 5.4. *Let $\{\mathbf{A}^{\ell} \boldsymbol{\phi}\}_{\ell=0}^{L-1}$ be a full-spark frame for \mathbb{C}^d , let $\alpha_1, \alpha_2 \in \mathbb{R}$ be real numbers with $\alpha_1 - \alpha_2 \notin \pi\mathbb{Z}$, and let $J \in \{0, \dots, d-2\}$. If $L \geq (d+1)^2/4(J+1) + d$, then every $\mathbf{x} \in \mathbb{C}^d$*

can be recovered from the samples

$$\left\{ |\langle \mathbf{x}, \mathbf{A}^\ell \boldsymbol{\phi} \rangle| \right\}_{\ell=0}^{L-1} \cup \left\{ |\langle \mathbf{x}, \mathbf{A}^\ell (\boldsymbol{\phi} + e^{i\alpha_k} \mathbf{A}^j \boldsymbol{\phi}) \rangle| \right\}_{\ell=0, k=1, j=1}^{L-2, 2, J+1}$$

up to global phase.

Proof. The difference to the proof of Theorem 5.3 is that we may here jump over J consecutive zeros while calculating the relative phases. Thus, if m measurements $|\langle \mathbf{x}, \mathbf{A}^\ell \boldsymbol{\phi} \rangle|$ are zero, the worst case scenario is that $d - m - J - 1$ consecutive non-zero measurements are followed by $J + 1$ zeros. Repeating this pattern $\lfloor m/(J+1) \rfloor$ times, and placing the remaining $m \bmod (J + 1) \leq m$ zeros and $d - m$ non-zeros at the end – so at most d elements, we require at most

$$\left\lfloor \frac{m}{J+1} \right\rfloor (d - m) + (d - m) + m \bmod (J + 1) \leq \frac{m}{J+1} (d - m) + d$$

measurements to transfer the relative phases far enough to recover \mathbf{x} . The maximum on the right-hand side is attained at $m := \lceil (d+1)/2 \rceil$ for odd d and $m := d/2$ for even d , which finishes the proof. \square

REFERENCES

- [1] A. C. Aitken. *Determinants and Matrices*. University Mathematical Texts. Oliver and Boyd, Edinburgh, 3rd edition, 1944.
- [2] R. Alaifari, I. Daubechies, P. Grohs, and R. Yin. Stable phase retrieval in infinite dimensions. *Found Comput Math*, 19(4):869–900, 2019.
- [3] A. Aldroubi, C. Cabrelli, A. F. Cakmak, U. Molter, and A. Petrosyan. Iterative actions of normal operators. *J Funct Anal*, 272(3):1121–1146, 2017.
- [4] A. Aldroubi, C. Cabrelli, U. Molter, and S. Tang. Dynamical sampling. *Appl Comput Harmon Anal*, 42(3):378–401, 2017.
- [5] A. Aldroubi, L. Huang, and A. Petrosyan. Frames induced by the action of continuous powers of an operator. *J Math Anal Appl*, 478(2):1059–1084, 2019.
- [6] A. Aldroubi and I. Krishtal. Krylov subspace methods in dynamical sampling. *Sampl Theory Signal Image Process*, 15:9–20, 2016.
- [7] A. Aldroubi, I. Krishtal, and S. Tang. Phase retrieval of evolving signals from space-time samples. In *Proceedings of the SampTA 2017*, pages 46–49, 2017.
- [8] A. Aldroubi, I. Krishtal, and S. Tang. Phaseless reconstruction from space-time samples. *Appl Comput Harmon Anal*, 48(1):395–414, 2020.
- [9] A. Aldroubi and A. Petrosyan. Dynamical sampling and systems from iterative actions of operators. In *Frames and Other Bases in Abstract and Function Spaces*, chapter 2, pages 15–26. Birkhäuser, Cham, 2017.
- [10] B. Alexeev, A. S. Bandeira, M. Fickus, and D. G. Mixon. Phase retrieval with polarization. *SIAM J Imaging Sci*, 7(1):35–66, 2014.
- [11] B. Alexeev, J. Cahill, and D. G. Mixon. Full spark frames. *J Fourier Anal Appl*, 18(6):1167–1194, 2012.
- [12] R. Balan, B. G. Bodmann, P. G. Casazza, and D. Edidin. Painless reconstruction from magnitudes of frame coefficients. *J Fourier Anal Appl*, 15(4):488–501, 2009.

- [13] R. Balan, P. Casazza, and D. Edidin. On signal reconstruction without phase. *Appl Comput Harmon Anal*, 20(3):345–356, 2006.
 - [14] A. S. Bandeira, J. Cahill, D. G. Mixon, and A. A. Nelson. Saving phase: injectivity and stability for phase retrieval. *Appl Comput Harmon Anal*, 37(1):106–125, 2014.
 - [15] R. Beinert. One-dimensional phase retrieval with additional interference measurements. *Results Math*, 72(1):1–24, 2017.
 - [16] R. Beinert and G. Plonka. Ambiguities in one-dimensional discrete phase retrieval from Fourier magnitudes. *J Fourier Anal Appl*, 21(6):169–1198, 2015.
 - [17] R. Beinert and G. Plonka. One-dimensional discrete-time phase retrieval. In *Nanoscale Photonic Imaging*, Nanoscale Photonic Imaging, chapter 24, pages 603–627. Springer, Cham, 2020.
 - [18] T. Bendory, R. Beinert, and Y. C. Eldar. Fourier phase retrieval: uniqueness and algorithms. In *Compressed Sensing and its Applications*, Applied and Numerical Harmonic Analysis, chapter 2, pages 55–91. Birkhäuser, Cham, 2017.
 - [19] Y. M. Bruck and L. G. Sodin. On the ambiguity of the image reconstruction problem. *Opt Commun*, 30(3):304–308, September 1979.
 - [20] C. Cabrelli, U. Molter, V. Paternostro, and F. Philipp. Dynamical sampling on finite index sets. *J Anal Math*, 140(2):637–667, 2020.
 - [21] E. J. Candès, Y. C. Eldar, T. Strohmer, and V. Voroninski. Phase retrieval via matrix completion. *SIAM J Imaging Sci*, 6(1):199–225, 2013.
 - [22] E. J. Candès, T. Strohmer, and V. Voroninski. PhaseLift: exact and stable signal recovery from magnitude measurements via convex programming. *Comm Pure Appl Math*, 66(8):1241–1274, 2013.
 - [23] O. Christensen. *An introduction to frames and Riesz bases*. Springer, 2016.
 - [24] O. Christensen and M. Hasannasab. Operator representations of frames: boundedness, duality, and stability. *Integral Equations Operator Theory*, 88(4):483–499, 2017.
 - [25] O. Christensen, M. Hasannasab, and F. Philipp. Frame properties of operator orbits. *Math Nachr*, 293(1):52–66, 2020.
 - [26] O. Christensen, M. Hasannasab, and E. Rashidi. Dynamical sampling and frame representations with bounded operators. *J Math Anal Appl*, 463(2):634–644, 2018.
 - [27] J. C. Dainty and J. R. Fienup. Phase retrieval and image reconstruction for astronomy. In *Image Recovery: Theory and Application*, chapter 7, pages 231–275. Academic Press, Orlando (Florida), 1987.
 - [28] S. Delvaux and M. Van Barel. Rank-deficient submatrices of Fourier matrices. *Linear Algebra Appl*, 429(7):1587–1605, 2008.
 - [29] P. Grohs, S. Koppensteiner, and M. Rathmair. The mathematics of phase retrieval. *arXiv preprint arXiv:1901.07911*, 2019.
 - [30] H. A. Hauptman. The phase problem of x-ray crystallography. *Rep Prog Phys*, 54(11):1427–1454, November 1991.
 - [31] D. Kalman. The generalized Vandermonde matrix. *Math Mag*, 57(1):15–21, 1984.
 - [32] W. Kim and M. H. Hayes. The phase retrieval problem in x-ray crystallography. In *Proceedings of the ICASSP 91*, volume 3, pages 1765–1768, 1991.
 - [33] M. V. Klivanov and V. G. Kamburg. Uniqueness of a one-dimensional phase retrieval problem. *Inverse Problems*, 30(7):075004(10), July 2014.
 - [34] M. V. Klivanov, P. E. Sacks, and A. V. Tikhonravov. The phase retrieval problem. *Inverse Problems*, 11(1):1–28, 1995.
-

-
- [35] Y. M. Lu and M. Vetterli. Spatial super-resolution of a diffusion field by temporal oversampling in sensor networks. In *Proceedings of the ICASSP 2009*, pages 2249–2252, 2009.
- [36] I. G. Macdonald. Schur functions: theme and variations. In *Séminaire Lotharingien de Combinatoire (Saint-Nabor, 1992)*, volume 498 of *Publ Inst Rech Math Av*, pages 5–39. Univ Louis Pasteur, Strasbourg, 1992.
- [37] I. G. Macdonald. *Symmetric functions and Hall polynomials*. Oxford Mathematical Monographs. Oxford University Press, Oxford, 2nd edition, 1995.
- [38] R. D. Malikiosis and V. Oussa. Full spark frames in the orbit of a representation. *Appl Comput Harmon Anal*, 49(3):791–814, 2020.
- [39] R. P. Millane. Phase retrieval in crystallography and optics. *J Opt Soc Amer A*, 7(3):394–411, 1990.
- [40] A. L. Patterson. A Fourier series method for the determination of the components of interatomic distances in crystals. *Phys Rev*, 46(5):372, 1934.
- [41] F. Philipp. Bessel orbits of normal operators. *J Math Anal Appl*, 448(2):767–785, 2017.
- [42] J. Ranieri, A. Chebira, Y. M. Lu, and M. Vetterli. Sampling and reconstructing diffusion fields with localized sources. In *Proceedings of the ICASSP 2011*, pages 4016–4019, 2011.
- [43] B. Seifert, H. Stolz, M. Donatelli, D. Langemann, and M. Tasche. Multilevel Gauss-Newton methods for phase retrieval problems. *J Phys A: Math Gen*, 39(16):4191–4206, 2006.
- [44] B. Seifert, H. Stolz, and M. Tasche. Nontrivial ambiguities for blind frequency-resolved optical gating and the problem of uniqueness. *J Opt Soc Am B*, 21(5):1089–1097, May 2004.
- [45] Y. Shechtman, Y. C. Eldar, O. Cohen, H. N. Chapman, J. Miao, and M. Segev. Phase retrieval with application to optical imaging: A contemporary overview. *IEEE Signal Process Mag*, 32(3):87–109, 2015.
- [46] P. van Hove, M. H. Hayes, J. S. Lim, and A. V. Oppenheim. Signal reconstruction from signed Fourier transform magnitude. *IEEE Trans Acoust Speech Signal Process*, ASSP-31(5):1286–1293, October 1983.
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