Ambiguities in one-dimensional discrete phase retrieval from FOURIER magnitudes

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based on joint work with

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Formulation of the problem

Problem (Phase retrieval)

Recover the unknown complex-valued signal

$$\boldsymbol{x} := (x[n])_{n \in \mathbb{Z}}$$

with finite support from its FOURIER *intensities*

$$|\hat{x}(\omega)|^2 .$$

Definition (Discrete FOURIER **transform)**

$$\hat{x}(\omega) := \sum_{n \in \mathbb{Z}} x[n] \, \mathrm{e}^{-\mathrm{i}\omega n} \qquad (\omega \in [-\pi, \pi)) .$$

Trivial ambiguities

Example

Let $\boldsymbol{x}$ be a complex-valued signal. Then

- the rotated signal

$$(y[n]) := \left(\mathrm{e}^{\mathrm{i}\alpha} x[n]\right),$$

- the shifted signal

$$(y[n]) := (x[n - n_0]),$$

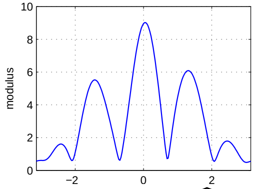
- the reflected conjugated signal

$$(y[n]) := \left(\overline{x[-n]}\right)$$

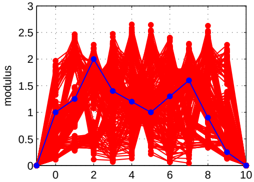
have the same Fourier intensities $|\widehat{x}(\omega)|^2$.

Non-trivial ambiguities

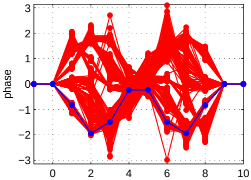
Example



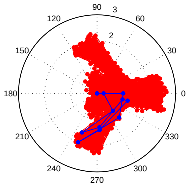
FOURIER intensities: $|\hat{x}(\omega)|$



Modulus: $|x[n]|$



Phase: $\arg(x[n])$



Polar plot of $\boldsymbol{x}$

The autocorrelation signal and function

- Autocorrelation signal

$$a[n] := \sum_{k\in\mathbb{Z}} x[k]\,\overline{x[k+n]} \qquad (n \in \mathbb{Z})\,.$$

- Autocorrelation function

$$A(\omega) := |\hat{x}(\omega)|^2 = \sum_{n\in\mathbb{Z}}\sum_{k\in\mathbb{Z}} x[n]\,\overline{x[k]}\,\mathrm{e}^{-\mathrm{i}\omega(n-k)} = \sum_{n\in\mathbb{Z}} a[n]\,\mathrm{e}^{-\mathrm{i}\omega n}.$$

Equivalent problem

Find the trigonometric polynomials $B(\omega)$ which satisfy

$$|B(\omega)|^2 = A(\omega).$$

The associated polynomial

(Fejér, Riesz [1916])

- Consider the complex polynomial $P_A(z)$ defined by

$$P_A\left(e^{-i\omega}\right) = e^{-i\omega(N-1)} A(\omega)$$

for a given autocorrelation function A, i.e.

$$P_A(z) := \sum_{n=0}^{2N-2} a[n-N+1]\, z^n \quad \text{with} \quad a[-n] = \overline{a[n]}.$$

- Obviously, we have

$$\left|P_A\left(e^{-i\omega}\right)\right| = A(\omega).$$

- Let γ_j be the roots of P_A.

Factorization of the associated polynomial
(Fejér, Riesz [1916])

- The roots occur in pairs $(\gamma_j, \overline{\gamma}_j^{-1})$:

$$P_A\left(\overline{\gamma}_j^{-1}\right) = \sum_{n=0}^{2N-2} a[n-N+1]\,\overline{\gamma}_j^{-n} = \sum_{n=0}^{2N-2} \overline{a[N-1-n]}\,\overline{\gamma}_j^{-n}$$

$$= \sum_{n=0}^{2N-2} \overline{a[n-N+1]}\,\overline{\gamma}_j^{-2N+2+n} = \overline{\gamma}_j^{-2N+2}\,\overline{P_A(\gamma_j)} = 0.$$

- P_A has the factorization

$$P_A(z) = a[N-1] \prod_{j=1}^{N-1} (z-\gamma_j)\left(z-\overline{\gamma}_j^{-1}\right).$$

Factorization of the autocorrelation function
(Fejér, Riesz [1916])

- Observe that

$$\left|\left(\mathrm{e}^{-\mathrm{i}\omega} - \gamma_j\right)\left(\mathrm{e}^{-\mathrm{i}\omega} - \overline{\gamma}_j^{\,-1}\right)\right| = \left|\gamma_j\right|^{-1}\left|\mathrm{e}^{-\mathrm{i}\omega} - \gamma_j\right|^2 .$$

- A has the factorization

$$\begin{aligned} A(\omega) &= \left|P_A\left(\mathrm{e}^{-\mathrm{i}\omega}\right)\right| \\ &= |a[N-1]| \prod_{j=1}^{N-1} \left|\gamma_j\right|^{-1} \left|\prod_{j=1}^{N-1}\left(\mathrm{e}^{-\mathrm{i}\omega} - \gamma_j\right)\right|^2 \\ &= |B(\omega)|^2 . \end{aligned}$$

Characterization of all ambiguities

(Real case: Bruck, Sodin [1979])

Theorem (B., Plonka [2014])

Let $A(\omega)$ be a non-negative trigonometric polynomial. Then the problem

$$|B(\omega)|^2 = A(\omega)$$

has at least one solution.
Furthermore, each solution can be written in the form

$$B(\omega) = \mathrm{e}^{\mathrm{i}\alpha + \mathrm{i}\omega n_0} \left[|a[N-1]| \prod_{j=1}^{N-1} \left|\beta_j\right|^{-1} \right]^{\frac{1}{2}} \prod_{j=1}^{N-1} \left(\mathrm{e}^{-\mathrm{i}\omega} - \beta_j \right)$$

where β_j can be chosen from the zero pairs $(\gamma_j, \overline{\gamma}_j^{-1})$ of the associated polynomial P_A.

Characterization by convolution

Definition (Convolution of signals)

$$(x_1 * x_2)[n] := \sum_{k \in \mathbb{Z}} x_1[k]\, x_2[n-k].$$

Theorem (B., Plonka [2014])

Let $|\hat{x}(\omega)|^2$ be given. Further, let $\boldsymbol{x}_1$ and $\boldsymbol{x}_2$ be two finite signals with

$$\boldsymbol{x} = \boldsymbol{x}_1 * \boldsymbol{x}_2.$$

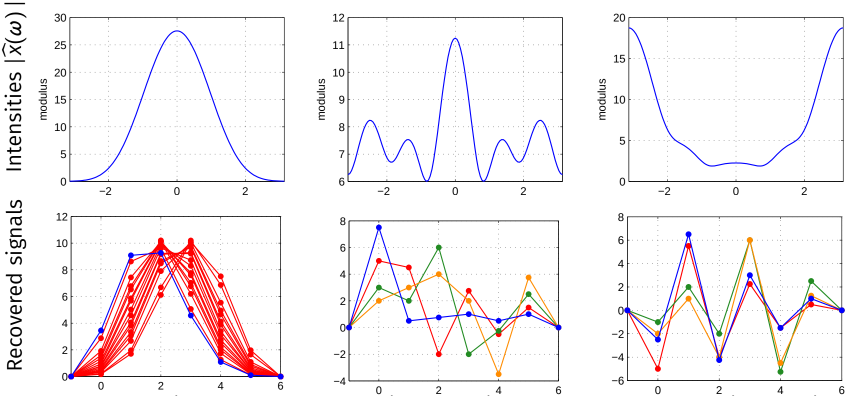
Then

$$\boldsymbol{y} := \mathrm{e}^{\mathrm{i}\alpha} \left(\overline{x_1[-\cdot]}\right) * (x_2[\cdot - n_0])$$

has the same Fourier intensities. Moreover, all signals with Fourier intensities $|\hat{x}(\omega)|^2$ have such a representation.

Positivity of the real signal

Example



Full solution set | Unique solution | No non-negative solution

Given moduli of the signal

(Seifert, Stolz, Donatelli, Langemann, Tasche [2006]; Langemann, Tasche [2008])

- Let the moduli $|x[n]|^2$ of the signal $\boldsymbol{x}$ be additionally given.
- Assume that $|x[N-1]|$ is given and $\widetilde{\boldsymbol{x}}$ is a further solution with $\widetilde{\beta}_j = \overline{\beta}_j^{-1}$ for $j = 1, \ldots, L$ and $\widetilde{\beta}_j = \beta_j$ else .
- The zeros have to fulfil the condition

$$\prod_{j=1}^{L} \left|\beta_j\right|^2 - 1 = 0.$$

Theorem (Beinert, Plonka [2014])

Almost every signal $\boldsymbol{x}$ can be uniquely recovered from the Fourier intensities $|\widehat{x}(\omega)|^2$ and the moduli $|x[n]|^2$ up to rotations.

Interference with a known signal

(Real Case: KIM, HAYES [1990])

- Let $\boldsymbol{h}$ be a known finite length signal.
- Assume that the FOURIER intensities $|\widehat{y}(\omega)|^2$ of

$$y[n] = x[n] + h[n] \qquad (n \in \mathbb{Z})$$

  are also given.
- In some cases, e.g. if $\boldsymbol{h}$ is a DIRAC signal or has linear phase, the phase retrieval problem is uniquely solvable up one trivial ambiguity.

Interference with an unknown signal

(Real case: Kim, Hayes [1993]; Complex case: Raz, Dudovich, Nadler [2013])

Theorem (B., Plonka [2014])

Let $\mathbf{x}$ *and* $\mathbf{h}$ *be two complex signals with finite support and assume that the factorizations of their symbols*

$$\widehat{x}(\omega) = \mathrm{e}^{\mathrm{i}\omega n_1}\, x[N_1 - 1] \prod_{j=1}^{N_1-1} \left(\mathrm{e}^{-\mathrm{i}\omega} - \eta_j\right),$$

$$\widehat{h}(\omega) = \mathrm{e}^{\mathrm{i}\omega n_2}\, h[N_2 - 1] \prod_{j=1}^{N_2-1} \left(\mathrm{e}^{-\mathrm{i}\omega} - \gamma_j\right)$$

have no common non-zero roots. Then $\mathbf{x}$ *and* $\mathbf{h}$ *can be uniquely recovered from* $|\widehat{x}(\omega)|^2$, $|\widehat{h}(\omega)|^2$ *and* $|\widehat{x}(\omega) + \widehat{h}(\omega)|^2$ *up to trivial ambiguities.*

Summary

- Complete characterization of the ambiguities in the one-dimensional phase retrieval problem.
- Quality of a priori conditions and additional data.

Outlook

- Phase retrieval in higher dimension.
- Ambiguities in the FRESNEL regime.

Thank you for your attention.

R. Beinert, G. Plonka. Ambiguities in one-dimensional discrete phase retrieval from Fourier magnitudes. *J. Fourier Anal. Appl.* to appear.