Ambiguities in one-dimensional phase retrieval from Fourier magnitudes

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Phase retrieval problem

Formulation of the problem

Problem (Phase retrieval)

Recover the unknown complex-valued signal

$$x := (x[n])_{n\in\mathbb{Z}}$$

with finite support from the FOURIER *intensity*

$$|\hat{x}(\omega)| \qquad (\omega \in \mathbb{R}).$$

Definition (Discrete-time FOURIER transform)

$$\hat{x}(\omega) := \sum_{n\in\mathbb{Z}} x[n]\, \mathrm{e}^{-\mathrm{i}\omega n} \qquad (\omega \in \mathbb{R})$$

Trivial ambiguities

Example

Let x be a complex-valued signal. Then

- the rotated signal

$$(y[n]) := (\mathrm{e}^{\mathrm{i}\alpha}\, x[n]),$$

- the shifted signal

$$(y[n]) := (x[n - n_0]),$$

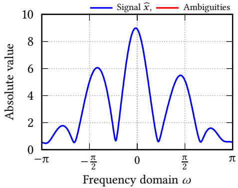
- the reflected, conjugated signal

$$(y[n]) := \left(\overline{x[-n]}\right)$$

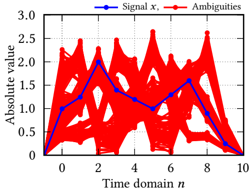
have the same Fourier intensity $|\widehat{x}|$.

Non-trivial ambiguities

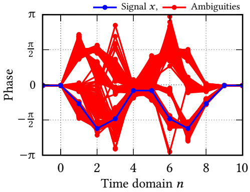
Example



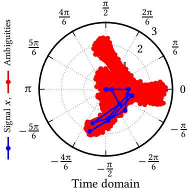
Fourier intensity: $|\widehat{x}(\omega)|$



Absolute value: $|x[n]|$



Phase: $\arg(x[n])$



Polar representation: $x[n]$

Characterizing the solution set

Autocorrelation signal and function

Definition (Autocorrelation signal)

$$a[n] := \sum_{k \in \mathbb{Z}} \overline{x[k]}\, x[k+n] \qquad (n \in \mathbb{Z}).$$

- The autocorrelation signal is conjugate symmetric, i.e.

$$\overline{a[-n]} = \sum_{k \in \mathbb{Z}} x[k]\, \overline{x[k-n]} = \sum_{k \in \mathbb{Z}} x[k+n]\, \overline{x[k]} = a[n] \quad (n \in \mathbb{Z}).$$

Definition (Autocorrelation function)

$$A(\omega) := \sum_{n \in \mathbb{Z}} a[n]\, \mathrm{e}^{-\mathrm{i}\omega n} = \sum_{n=-N+1}^{N-1} a[n]\, \mathrm{e}^{-\mathrm{i}\omega n}.$$

- The autocorrelation function is a non-negative trigonometric polynomial of degree $N-1$.

Phase retrieval in the frequency domain

- Relationship to the FOURIER transform:

$$|\widehat{x}(\omega)|^2 = \Big(\sum_{n\in\mathbb{Z}} x[n]\,\mathrm{e}^{-\mathrm{i}\omega n}\Big)\Big(\sum_{k\in\mathbb{Z}} \overline{x[k]}\,\mathrm{e}^{\mathrm{i}\omega k}\Big)$$

$$= \sum_{n\in\mathbb{Z}} \sum_{k\in\mathbb{Z}} x[k+n]\,\overline{x[k]}\,\mathrm{e}^{-\mathrm{i}\omega n} = A(\omega).$$

Equivalent problem

Find a trigonometric polynomial B such that

$$|B(\omega)|^2 = A(\omega).$$

Associate polynomial

Definition (associate polynomial)

$$P_A(e^{-i\omega}) = e^{-i\omega(N-1)}\,A(\omega)$$

- The algebraic polynomial P_A is thus defined by

$$P_A(z) := \sum_{n=0}^{2N-2} a[n-N+1]\,z^n \qquad \text{with} \qquad a[-n] = \overline{a[n]}.$$

- Obviously, we have

$$A(\omega) = \left|P_A(e^{-i\omega})\right|.$$

- P_A has the factorization

$$P_A(z) = a[N-1]\prod_{j=1}^{N-1}(z-\gamma_j)(z-\overline{\gamma}_j^{\,-1}).$$

Factorization of the autocorrelation function

- For $z := \mathrm{e}^{-\mathrm{i}\omega}$, the absolute value of the linear factors is

$$\left|\left(\mathrm{e}^{-\mathrm{i}\omega} - \gamma_j\right)\left(\mathrm{e}^{-\mathrm{i}\omega} - \overline{\gamma}_j^{\,-1}\right)\right| = \left|\mathrm{e}^{-\mathrm{i}\omega} - \gamma_j\right|\left|\overline{\gamma}_j^{\,-1}\right|\left|\overline{\gamma}_j - \mathrm{e}^{\mathrm{i}\omega}\right|$$
$$= \left|\gamma_j\right|^{-1}\left|\mathrm{e}^{-\mathrm{i}\omega} - \gamma_j\right|^2.$$

- A has the factorization

$$A(\omega) = \left|P_A(\mathrm{e}^{-\mathrm{i}\omega})\right| = |a[N-1]| \prod_{j=1}^{N-1} \left|\left(\mathrm{e}^{-\mathrm{i}\omega} - \gamma_j\right)\left(\mathrm{e}^{-\mathrm{i}\omega} - \overline{\gamma}_j^{\,-1}\right)\right|$$

$$= |a[N-1]| \prod_{j=1}^{N-1} \left|\beta_j\right|^{-1} \left|\prod_{j=1}^{N-1} (\mathrm{e}^{-\mathrm{i}\omega} - \beta_j)\right|^2 = |B(\omega)|^2$$

with $\beta_j \in (\gamma_j, \overline{\gamma}_j^{\,-1})$.

Non-trivial ambiguities

Real case: BRUCK, SODIN [1979]

Theorem (BEINERT, PLONKA [2015])

Let A be a non-negative trigonometric polynomial. Then the problem

$$\left|B(\omega)\right|^2 = A(\omega)$$

has at least one solution. Every solution has a representation of the form

$$B(\omega) = \mathrm{e}^{\mathrm{i}\alpha + \mathrm{i}\omega n_0} \sqrt{|a[N-1]| \prod_{j=1}^{N-1} \left|\beta_j\right|^{-1}} \cdot \prod_{j=1}^{N-1} (\mathrm{e}^{-\mathrm{i}\omega} - \beta_j),$$

where β_j can be chosen from the zero pair $(\gamma_j, \overline{\gamma}_j^{\,-1})$ of the associated polynomial P_A.

Representation of the ambiguities in time domain

Definition (Convolution of signals)

$$(x_1 * x_2)[n] := \sum_{k \in \mathbb{Z}} x_1[k]\, x_2[n-k].$$

Theorem (Beinert, Plonka [2015])

Let x be a signal with finite support and factorization

$$x = x_1 * x_2.$$

Then the signal

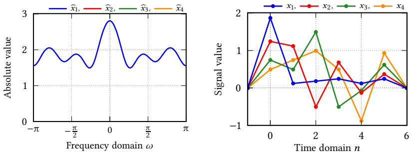
$$y := \mathrm{e}^{\mathrm{i}\alpha} \left(\overline{x_1[-\cdot]}\right) * \left(x_2[\cdot - n_0]\right)$$

has the same Fourier intensity $|\widehat{x}|$ *and all signals with the Fourier intensity* $|\widehat{x}|$ *can be represented in this manner.*

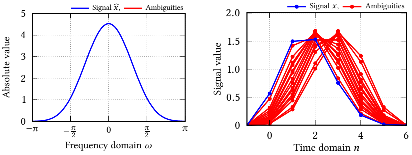
Ensuring uniqueness

Phase retrieval of non-negative signals

Example



Unique non-negative solution



Full non-negative solution set

Theorem (Beinert [2015])

The sets of non-negative signals with support length $N > 3$ that

- *can be recovered uniquely up to reflection*
- *cannot be recovered uniquely up to reflection*

from their Fourier intensities are unbounded sets of infinite Lebesgue measure.

Knowledge of additional moduli

- Recover x from $|\widehat{x}|$ and $|x[N-1-\ell]|$ for an ℓ.
- Assume that there exist two non-trivial solutions x and $\widetilde{x}$.
- For $|x[N-1-\ell]| = |\widetilde{x}[N-1-\ell]|$, Vieta's formulae yield the condition

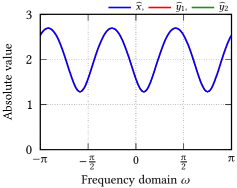
$$\prod_{j=1}^{N-1} \left|\beta_j\right|^{-\frac{1}{2}} \cdot \left| \sum_{1 \le k_1 < \cdots < k_\ell \le N-1} \beta_{k_1} \cdots \beta_{k_\ell} \right| = \prod_{j=1}^{N-1} \left|\widetilde{\beta}_j\right|^{-\frac{1}{2}} \cdot \left| \sum_{1 \le k_1 < \cdots < k_\ell \le N-1} \widetilde{\beta}_{k_1} \cdots \widetilde{\beta}_{k_\ell} \right|.$$

Theorem (Beinert, Plonka [2015])

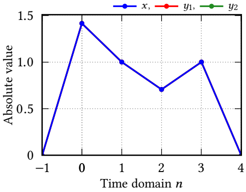
Almost every signal x can be recovered from $|\widehat{x}|$ and $|x[N-1-\ell]|$ for an arbitrary $\ell \neq {}^{(N-1)}/_2$ up to rotations, for $\ell = {}^{(N-1)}/_2$ up to reflection/conjugation and rotation.

Knowledge of additional moduli

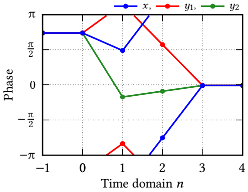
Example



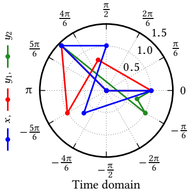
FOURIER intensity: $|\widehat{x}(\omega)|$



Absolute value: $|x[n]|$



Phase: $\arg(x[n])$



Polar representation: $x[n]$

Knowledge of additional phases

Theorem (Beinert [2015])

Let ℓ_1 and ℓ_2 two different integers in $\{0, \ldots, N-1\}$. Then almost every signal x can be uniquely recovered from $|\widehat{x}|$ and the phases

$$\arg x[N-1-\ell_1] \quad \text{and} \quad \arg x[N-1-\ell_2] \qquad (\ell_1 + \ell_2 \neq N-1).$$

For $\ell_1 + \ell_2 = N-1$, the recovery of the unknown signal is only unique up to reflection/conjugation, except for the case where the phase of both end points is given.

Interference with reference signal

Real case: Kim, Hayes [1993]
Complex case: Raz, Dudovich, Nadler [2013]

Theorem (Beinert, Plonka [2015])

Let x and h be complex-valued signals with finite support, and assume that the factorization of their symbols

$$\widehat{x}(\omega) = \mathrm{e}^{\mathrm{i}\omega n_1}\, x[N_1 - 1] \prod_{j=1}^{N_1-1} (\mathrm{e}^{-\mathrm{i}\omega} - \eta_j)$$

and

$$\widehat{h}(\omega) = \mathrm{e}^{\mathrm{i}\omega n_2}\, h[N_2 - 1] \prod_{j=1}^{N_2-1} (\mathrm{e}^{-\mathrm{i}\omega} - \gamma_j)$$

have no common zeros. Then x and h can be uniquely recovered from $|\widehat{x}(\omega)|$, $|\widehat{h}(\omega)|$ and $|\widehat{x}(\omega) + \widehat{h}(\omega)|$ up to common trivial ambiguities.

Interference with reference signal

Sketch of proof

- Assume there are two solutions $x[n]$, $h[n]$ and $\widetilde{x}[n]$, $\widetilde{h}[n]$.
- Use the factorization in the frequency domain:

$$\widehat{x}(\omega) = \mathrm{e}^{\mathrm{i}\omega n_1}\,\widehat{x}_1(\omega)\,\widehat{x}_2(\omega) \quad \text{and} \quad \widehat{\widetilde{x}}(\omega) = \mathrm{e}^{\mathrm{i}\alpha_1}\mathrm{e}^{\mathrm{i}\omega k_1}\,\widehat{x}_1(\omega)\,\overline{\widehat{x}_2(\omega)},$$

$$\widehat{h}(\omega) = \mathrm{e}^{\mathrm{i}\omega n_2}\,\widehat{h}_1(\omega)\,\widehat{h}_2(\omega) \quad \text{and} \quad \widehat{\widetilde{h}}(\omega) = \mathrm{e}^{\mathrm{i}\alpha_2}\mathrm{e}^{\mathrm{i}\omega k_2}\,\widehat{h}_1(\omega)\,\overline{\widehat{h}_2(\omega)}.$$

- Consider the identity

$$\left|\widehat{x}(\omega) + \widehat{h}(\omega)\right|^2 = \left|\widehat{\widetilde{x}}(\omega) + \widehat{\widetilde{h}}(\omega)\right|^2.$$

Interference with reference signal

Theorem (Beinert [2015])

Let f and h be complex-valued continuous-time signals with compact support, and assume that the factorization of their Laplace transforms

$$F(\zeta) = C_1\, \zeta^{m_1}\, \mathrm{e}^{\zeta \gamma_1} \prod_{j=1}^{\infty} \left(1 - \frac{\zeta}{\xi_j}\right) \mathrm{e}^{\frac{\zeta}{\xi_j}}$$

and

$$H(\zeta) = C_2\, \zeta^{m_2}\, \mathrm{e}^{\zeta \gamma_2} \prod_{j=1}^{\infty} \left(1 - \frac{\zeta}{\eta_j}\right) \mathrm{e}^{\frac{\zeta}{\eta_j}}$$

have no common zeros. Then f and h can be uniquely recovered from $|\widehat{f}(\omega)|$, $|\widehat{h}(\omega)|$ and $|\widehat{f}(\omega) + \widehat{h}(\omega)|$ up to common trivial ambiguities.

Summary/Outlook

- Characterization of the ambiguities in the one-dimensional discrete-time phase retrieval problem.
- Investigation of the quality of different a priori conditions and additional data.

- Phase retrieval in higher dimensions.
- Transferring further results between the discrete-time and continuous-time problem.
- Investigation and development of numerical algorithms.

Thank you for the attention.

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Appendix

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