Discrete Fourier phase retrieval: ambiguities and uniqueness guarantees

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Discrete-time phase retrieval

Problem (Discrete-time phase retrieval)

Recover the unknown, complex-valued, and discrete-time signal $x \colon \mathbb{Z} \to \mathbb{C}$ with finite support from its Fourier intensity

$$|\widehat{x}(\omega)| \qquad (\omega \in \mathbb{R}).$$

Definition (Discrete-time Fourier transform)

$$\widehat{x}(\omega) := \sum_{n \in \mathbb{Z}} x[n]\, \mathrm{e}^{-\mathrm{i}\omega n} \qquad (\omega \in \mathbb{R}).$$

Besides the true signal x, the phase retrieval problem is solved by

- the rotated signal $(\mathrm{e}^{\mathrm{i}\alpha}\, x[n])_{n \in \mathbb{Z}}$,
- the shifted signal $(x[n - n_0])_{n \in \mathbb{Z}}$,
- the reflected, conjugated signal $(\overline{x[-n]})_{n \in \mathbb{Z}}$.

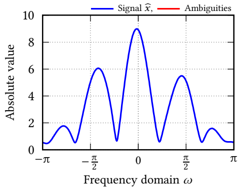
NON-TRIVIAL AMBIGUITIES

Definition (Trivial and non-trivial ambiguities)

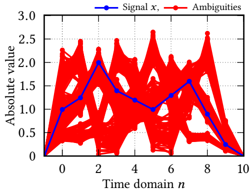
A *trivial ambiguity* is caused by rotation, shift, or reflection and conjugation.

All other occurring ambiguities are called *non-trivial*.

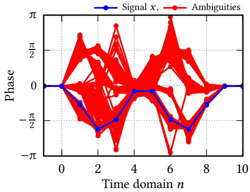
Example (Non-trivial ambiguities)



FOURIER intensity: $|\hat{x}(\omega)|$



Absolute value: $|x[n]|$



Phase: $\arg(x[n])$

Characterizing the solutions

Definition (Autocorrelation signal)

$$a[n] := \sum_{k\in\mathbb{Z}} \overline{x[k]}\, x[k+n] \qquad (n \in \mathbb{Z}).$$

- The autocorrelation signal is conjugate symmetric, i.e.

$$\overline{a[-n]} = \sum_{k\in\mathbb{Z}} x[k]\, \overline{x[k-n]} = \sum_{k\in\mathbb{Z}} x[k+n]\, \overline{x[k]} = a[n] \qquad (n \in \mathbb{Z}),$$

  and has the support $\{-N+1, \ldots, N-1\}$, where N denotes the support length of x.

Definition (Autocorrelation function)

$$A(\omega) := \sum_{n\in\mathbb{Z}} a[n]\, \mathrm{e}^{-\mathrm{i}\omega n} = \sum_{n=-N+1}^{N-1} a[n]\, \mathrm{e}^{-\mathrm{i}\omega n}.$$

- Relationship to the Fourier transform:

$$|\widehat{x}(\omega)|^2 = \Big(\sum_{n\in\mathbb{Z}} x[n]\,\mathrm{e}^{-\mathrm{i}\omega n}\Big)\Big(\sum_{k\in\mathbb{Z}} \overline{x[k]}\,\mathrm{e}^{\mathrm{i}\omega k}\Big)$$

$$= \sum_{n\in\mathbb{Z}}\sum_{k\in\mathbb{Z}} x[k+n]\,\overline{x[k]}\,\mathrm{e}^{-\mathrm{i}\omega n} = A(\omega).$$

- The autocorrelation function is a non-negative trigonometric polynomial of degree $N-1$.

Equivalent problem

Find a trigonometric polynomial B such that

$$|B(\omega)|^2 = A(\omega).$$

Definition (Associated polynomial)

$$P_A(e^{-i\omega}) = e^{-i\omega(N-1)}\,A(\omega).$$

- The algebraic polynomial P_A is thus defined by

$$P_A(z) := \sum_{n=0}^{2N-2} a[n-N+1]\,z^n \qquad \text{with} \qquad a[-n] = \overline{a[n]}.$$

- The zeros occur in pairs $(\gamma_j, \overline{\gamma}_j^{-1})$.
- P_A has the factorization

$$P_A(z) = a[N-1]\prod_{j=1}^{N-1}(z-\gamma_j)(z-\overline{\gamma}_j^{-1}).$$

- For $z := \mathrm{e}^{-\mathrm{i}\omega}$, the absolute value of the linear factors is

$$\left|\left(\mathrm{e}^{-\mathrm{i}\omega} - \gamma_j\right)\left(\mathrm{e}^{-\mathrm{i}\omega} - \overline{\gamma}_j^{\,-1}\right)\right| = |\overline{\gamma}_j|^{-1}\left|\mathrm{e}^{-\mathrm{i}\omega} - \gamma_j\right|\left|\overline{\gamma}_j - \mathrm{e}^{\mathrm{i}\omega}\right|$$

$$= |\gamma_j|^{-1}\left|\mathrm{e}^{-\mathrm{i}\omega} - \gamma_j\right|^2.$$

- A has the factorization

$$A(\omega) = |P_A(\mathrm{e}^{-\mathrm{i}\omega})| = |a[N-1]|\prod_{j=1}^{N-1}\left|\left(\mathrm{e}^{-\mathrm{i}\omega} - \gamma_j\right)\left(\mathrm{e}^{-\mathrm{i}\omega} - \overline{\gamma}_j^{\,-1}\right)\right|$$

$$= |a[N-1]|\prod_{j=1}^{N-1}|\beta_j|^{-1} \cdot \left|\prod_{j=1}^{N-1}(\mathrm{e}^{-\mathrm{i}\omega} - \beta_j)\right|^2 = |B(\omega)|^2$$

with $\beta_j \in (\gamma_j, \overline{\gamma}_j^{\,-1})$.

Theorem (Beinert, Plonka [2015])

Let A be a non-negative trigonometric polynomial. Then the problem

$$|B(\omega)|^2 = A(\omega)$$

has at least one solution. Every solution has a representation of the form

$$B(\omega) = \mathrm{e}^{\mathrm{i}\alpha + \mathrm{i}\omega n_0} \sqrt{|a[N-1]| \prod_{j=1}^{N-1} |\beta_j|^{-1}} \cdot \prod_{j=1}^{N-1} (\mathrm{e}^{-\mathrm{i}\omega} - \beta_j),$$

where β_j can be chosen from the zero pair $(\gamma_j, \overline{\gamma}_j^{-1})$ of the associated polynomial P_A.

- Each non-trivial solution is completely determined by its *corresponding zero set* $\{\beta_1, \ldots, \beta_{N-1}\}$.

Corollary

The number of non-trivial ambiguities may vary from 1 *up to* 2^{N-2}.

Proposition (Beinert [2015])

Let L *be the number of distinct zero pairs* $(\gamma_\ell, \overline{\gamma}_\ell^{-1})$ *of* P_A *not lying on the unit circle, and let* m_ℓ *be the multiplicity of these zero pairs. The corresponding phase retrieval problem has*

$$\left\lceil \frac{1}{2} \prod_{\ell=1}^{L} (m_\ell + 1) \right\rceil$$

non-trivial ambiguities.

Definition (Convolution of signals)

$$(x_1 * x_2)[n] := \sum_{k \in \mathbb{Z}} x_1[k]\, x_2[n-k].$$

Theorem (Beinert, Plonka [2015])

Let x be a signal with finite support and factorization

$$x = x_1 * x_2.$$

Then the signal

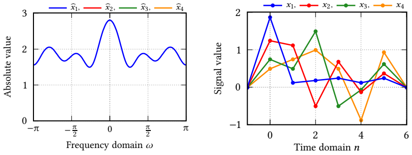
$$y := \mathrm{e}^{\mathrm{i}\alpha} \left(\overline{x_1[-\cdot]}\right) * (x_2[\cdot - n_0])$$

has the same Fourier intensity $|\widehat{x}|$ and all signals with the Fourier intensity $|\widehat{x}|$ can be represented in this manner.

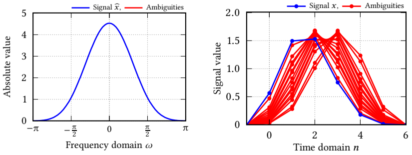
Non-negativity constraints

Phase retrieval of non-negative signals

Example



Unique non-negative solution



Full non-negative solution set

- The solutions of the phase retrieval problem have the form

$$\widehat{x}(\omega) = \mathrm{e}^{\mathrm{i}\omega n_0} \sqrt{|a[N-1]| \prod_{j=1}^{N-1} \left|\beta_j\right|^{-1}} \cdot \prod_{j=1}^{N-1} (\mathrm{e}^{-\mathrm{i}\omega} - \beta_j).$$

- A solution is non-negative if and only if all coefficients of

$$Q(z) := \prod_{j=1}^{N-1} (z - \beta_j)$$

are non-negative.

Lemma

Assume that $(\beta_{N-2}, \beta_{N-1})$ is a conjugate zero pair unequal to zero, and define

$$\sigma_n := (-1)^n \, S_n(\beta_1, \dots, \beta_{N-3}).$$

Then Q has non-negative coefficients if and only if β_{N-1} fulfils

$$\sigma_{n-2} |\beta_{N-1}|^2 - 2\sigma_{n-1} \Re \beta_{N-1} + \sigma_n \geq 0$$

for $n = 0, \dots, N-1$, and if σ_{N-3} is non-zero.

Definition (Elementary symmetric polynomials)

The *elementary symmetric polynomials* S_n are defined by

$$S_n(\beta_1, \dots, \beta_{N-3}) := \sum_{1 \leq k_1 < \dots < k_n \leq N-3} \beta_{k_1} \cdots \beta_{k_n} \qquad (n = 1, \dots, N-3)$$

as well as $S_0 := 1$ and $S_n := 0$ for $n < 0$ and $n > N-3$.

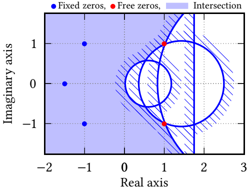
Corollary

Assume that the $\Re \beta_j < 0$ *for* $j = 1, \ldots, N-3$. *Then* Q *has non-negative coefficients if and only if*

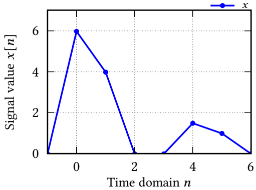
$$\Re \beta_{N-1} \leq {}^{\sigma_1}/_{2} \qquad \text{and} \qquad \left| \beta_{N-1} - \frac{\sigma_{n-1}}{\sigma_{n-2}} \right| \geq \frac{\sqrt{\sigma_{n-1}^2 - \sigma_n \sigma_{n-2}}}{\sigma_{n-2}}$$

for $n = 2, \ldots, N-2$ *whenever the radius is real.*

Example



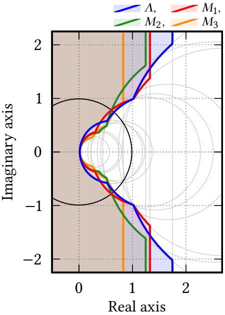
Restrictions on $(\beta_{N-2}, \beta_{N-1})$

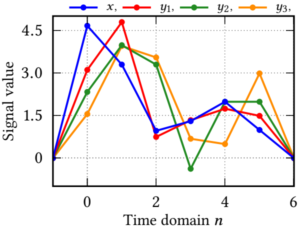


Corresponding signal

Example

Restriction on $(\beta_{N-2}, \beta_{N-1})$





Signals with $\beta_{N-1} = 3/4 + \mathrm{i}$

Theorem (Beinert [2017])

The sets of non-negative signals with support length $N > 3$ that

- *can/cannot be recovered uniquely up to reflection*

are unbounded sets of infinite Lebesgue measure.

Additional data in time domain

Knowledge of additional moduli

- Recover x from $|\hat{x}|$ and $|x[N-1-\ell]|$ for an ℓ within the fixed support $\{0, \ldots, N-1\}$.
- Assume that there exist two non-trivial solutions x and $\tilde{x}$.
- The Fourier transform of x can be written as

$$\hat{x}(\omega) = \mathrm{e}^{\mathrm{i}\alpha} \sqrt{|a[N-1]| \prod_{j=1}^{N-1} |\beta_j|^{-1}} \cdot \prod_{j=1}^{N-1} (\mathrm{e}^{-\mathrm{i}\omega} - \beta_j),$$

and $\hat{\tilde{x}}$ has an analogous representation with $\tilde{\beta}_j \in \{\beta_j, \overline{\beta}_j^{-1}\}$

- For $|x[N-1-\ell]| = |\tilde{x}[N-1-\ell]|$, Vieta's formulae yield

$$\prod_{j=1}^{N-1} |\beta_j|^{-\frac{1}{2}} \cdot \left| \sum_{1 \le k_1 < \cdots < k_\ell \le N-1} \beta_{k_1} \cdots \beta_{k_\ell} \right| = \prod_{j=1}^{N-1} |\tilde{\beta}_j|^{-\frac{1}{2}} \cdot \left| \sum_{1 \le k_1 < \cdots < k_\ell \le N-1} \tilde{\beta}_{k_1} \cdots \tilde{\beta}_{k_\ell} \right|,$$

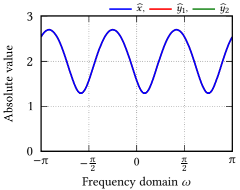
which can be written as a polynomial equation in $\Re \beta_j$ and $\Im \beta_j$.

Knowledge of additional moduli

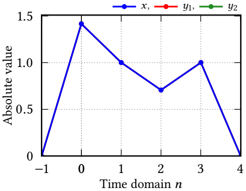
Theorem (Beinert, Plonka [2017])

Almost every signal x can be recovered from $|\widehat{x}|$ and $|x[N-1-\ell]|$ for an arbitrary $\ell \neq {}^{(N-1)}/_2$ up to rotations, for $\ell = {}^{(N-1)}/_2$ up to reflection/conjugation and rotation.

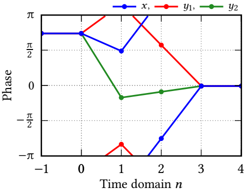
Example



Fourier intensity: $|\widehat{x}(\omega)|$



Absolute value: $|x[n]|$



Phase: $\arg(x[n])$

- Recover x from $|\widehat{x}|$, $\arg x[N-1-\ell_1]$, and $\arg x[N-1-\ell_2]$ for $\ell_1 \neq \ell_2$ within the fixed support $\{0, \ldots, N-1\}$.
- Assume that there exist two non-trivial solutions x and $\widetilde{x}$.
- Rewrite phase conditions into the equation

$$\Re\left[S_{\ell_1}(B)\right]\Im\left[\overline{S_{\ell_2}(\widetilde{B})}\, S_{\ell_2}(B)\, S_{\ell_1}(\widetilde{B})\right] - \Im\left[S_{\ell_1}(B)\right]\Re\left[\overline{S_{\ell_2}(\widetilde{B})}\, S_{\ell_2}(B)\, S_{\ell_1}(\widetilde{B})\right] = 0$$

with

$$B := \{\beta_1, \ldots, \beta_{N-1}\} \qquad \text{and} \qquad \widetilde{B} := \{\widetilde{\beta}_1, \ldots, \widetilde{\beta}_{N-1}\}.$$

Theorem (Beinert, Plonka [2017])

Let ℓ_1 and ℓ_2 two different integers in $\{0, \ldots, N-1\}$. Then almost every signal x can be uniquely recovered from $|\widehat{x}|$ and the phases

$$\arg x[N-1-\ell_1] \quad \text{and} \quad \arg x[N-1-\ell_2] \qquad (\ell_1 + \ell_2 \neq N-1).$$

For $\ell_1 + \ell_2 = N - 1$, the recovery of the unknown signal is only unique up to reflection/conjugation, except for the case where the phase of both end points is given.

Interference measurements

Interference with reference signal

Theorem (Beinert, Plonka [2015])

Let x and h be complex-valued signals with finite support, and assume that the factorization of their symbols

$$\widehat{x}(\omega) = e^{i\omega n_1}\, x[N_1 - 1] \prod_{j=1}^{N_1 - 1} (e^{-i\omega} - \eta_j)$$

and

$$\widehat{h}(\omega) = e^{i\omega n_2}\, h[N_2 - 1] \prod_{j=1}^{N_2 - 1} (e^{-i\omega} - \gamma_j)$$

have no common zeros. Then x and h can be uniquely recovered from $|\widehat{x}(\omega)|$, $|\widehat{h}(\omega)|$ *and* $|\widehat{x}(\omega) + \widehat{h}(\omega)|$ *up to common trivial ambiguities.*

Sketch of proof

- Assume there are two solutions $x[n]$, $h[n]$ and $\widetilde{x}[n]$, $\widetilde{h}[n]$.
- Use the factorization in the frequency domain:

$$\widehat{x}(\omega) = \mathrm{e}^{\mathrm{i}\omega n_1}\,\widehat{x}_1(\omega)\,\widehat{x}_2(\omega) \quad \text{and} \quad \widehat{\widetilde{x}}(\omega) = \mathrm{e}^{\mathrm{i}\alpha_1}\mathrm{e}^{\mathrm{i}\omega k_1}\,\widehat{x}_1(\omega)\,\overline{\widehat{x}_2(\omega)},$$

$$\widehat{h}(\omega) = \mathrm{e}^{\mathrm{i}\omega n_2}\,\widehat{h}_1(\omega)\,\widehat{h}_2(\omega) \quad \text{and} \quad \widehat{\widetilde{h}}(\omega) = \mathrm{e}^{\mathrm{i}\alpha_2}\mathrm{e}^{\mathrm{i}\omega k_2}\,\widehat{h}_1(\omega)\,\overline{\widehat{h}_2(\omega)}.$$

- Consider the identity

$$\left|\widehat{x}(\omega) + \widehat{h}(\omega)\right|^2 = \left|\widehat{\widetilde{x}}(\omega) + \widehat{\widetilde{h}}(\omega)\right|^2.$$

Sparsity constraints

- Recover x with support length N and sparsity $M := |\operatorname{supp} x|$ from its Fourier intensity $|\widehat{x}|$.
- Assume that the differences $n - m$ with $n \neq m$ and $n, m \in \operatorname{supp} x$ are pairwise distinct.
- Apply Prony's method to recover $n - m$ and $x[n]\,\overline{x[m]}$ from $|\widehat{x}|$ for all $n \neq m$ and $n, m \in \operatorname{supp} x$.

Theorem (Beinert, Plonka [2017])

Let x be a sparse signal. If

(i) *the differences $n - m$ differ pairwise for $n, m \in \operatorname{supp} x$, $n \neq m$*

(ii) *the coefficients satisfy $|x[n_{\min}]| \neq |x[n_{\max}]|$*

then x can be uniquely recovered from $3/2\, M(M-1) + 1$ equispaced measurements of $|\widehat{x}|$ up to trivial ambiguities.

Summary/Outlook

- Characterization of the ambiguities in the one-dimensional discrete-time phase retrieval problem.
- Investigation of the quality of different a priori conditions and additional data:
    - Non-negativity constraints,
    - Additional signal information in time domain,
    - Interference measurements,
    - Sparsity constraints.

- Phase retrieval in higher dimensions.
- Transferring further results between the discrete-time and continuous-time problem.
- Investigation and development of numerical algorithms.
- Consider further phase retrieval problems with respect to the Fresnel or short-time Fourier transform.

Thank you for the attention.

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