Regularization of bilinear and quadratic inverse problems by tensorial lifting

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The deautoconvolution problem

Problem (Deautoconvolution)

Recover the unknown complex-valued signal $u \in L^2_{\mathbb{C}}([0,1])$ *from its kernel-based autoconvolution*

$$A_k[u](t) \coloneqq \int_{-\infty}^{\infty} k(s,t)\, u(s)\, u(t-s)\, \mathrm{d}s \qquad (t \in \mathbb{R}),$$

where $k \in L^\infty_{\mathbb{C}}([0,1] \times [0,2])$ *is an appropriate kernel function.*

Proposition (Ambiguities deautoconvolution)

Let $g^\dagger$ *be in* $L_{\mathbb{C}^2([0,2])}$, *and let* $k \equiv 1$ *be the trivial kernel. If* $u \in L^2_{\mathbb{C}}([0,1])$ *is a solution of the deautoconvolution problem* $\mathcal{A}_k[u] = g^\dagger$, *then* u *is uniquely determined up to a global sign.*

Local ill-posedness

- The deautoconvolution problem is locally ill-posed everywhere.
- For every $u \in L^2_{\mathbb{C}}([0,1])$ and every $r > 0$, there exists a sequence $(u_n)_{n\in\mathbb{N}}$ with $\| u_n - u \| < r$ such that

$$\| \mathcal{A}_k[u_n] - \mathcal{A}_k[u] \| \to 0 \quad \text{but} \quad \| u_n - u \| \nrightarrow 0 \quad \text{as} \quad n \to \infty.$$

D. Gerth et al.: *Regularization of an autoconvolution problem in ultrashort laser pulse characterization.* (2013)

Problem (Tikhonov regularization)

Minimize the Tikhonov functional

$$J_\alpha(u) := \| \mathcal{A}_k[u] - g^\delta \|^2 + \alpha \| u \|^2$$

- Most of the requirements of the non-linear regularization theory in order to establish convergence rates are not fulfilled.

S. Bürger, B. Hofmann: *About a deficit in low-order convergence rates on the example of autoconvolution.* (2015)

S.W. Anzengruber et al.: *Variational regularization of complex deautoconvolution and phase retrieval in ultrashort laser pulse characterization.* (2016)

Tensor products

- Let X_1 and X_2 be some Banach spaces.
- The tensor $u \otimes v$ with $u \in X_1$ and $v \in X_2$ is a linear operator acting on the space $B(X_1 \times X_2)$ of all bilinear forms by

$$(u \otimes v)(A) = A(u, v) \qquad (A \in B(X_1 \times X_2)).$$

Definition (Tensor product)

The tensor product $X_1 \otimes X_2$ consists of all linear combinations

$$w = \sum_{n=1}^{N} \lambda_n \, u_n \otimes v_n$$

with $u_n \in X_1$, $v_n \in X_2$, $\lambda_n \in \mathbb{R}$, and $N \in \mathbb{N}$.

- The mapping $\otimes\colon (u, v) \mapsto u \otimes v$ is itself bilinear:
  (i) $(u_1 + u_2) \otimes v = u_1 \otimes v + u_2 \otimes v$,
  (ii) $u \otimes (v_1 + v_2) = u \otimes v_1 + u \otimes v_2$,
  (iii) $\lambda(u \otimes v) = (\lambda u) \otimes v = u \otimes (\lambda v)$, and
  (iv) $0 \otimes v = u \otimes 0 = 0$.

Definition (Projective norm)

The *projective norm* $\|\cdot\|_\pi$ on $X_1 \otimes X_2$ is defined by

$$\| w \|_\pi := \inf\left\{ \sum_{n=1}^{N} \| u_n \| \| v_n \| : w = \sum_{n=1}^{N} u_n \otimes v_n \right\}.$$

Definition (Projective tensor product)

The *projective tensor product* $X_1 \otimes_\pi X_2$ is the completion of $X_1 \otimes X_2$ with respect to $\|\cdot\|_\pi$.

Proposition (Lifting of bilinear operator)

Let X_1, X_2, and Y be Banach spaces, and let $A\colon X_1 \times X_2 \to Y$ be a bounded bilinear operator. Then there exists a unique bounded linear operator

$$\breve{A}\colon X_1 \otimes_\pi X_2 \to Y \qquad \text{so that} \qquad A(u,v) = \breve{A}(u \otimes v).$$

- The dual space of the projective tensor product is

$$(X_1 \otimes_\pi X_2)^* \simeq \mathcal{B}(X_1 \times X_2).$$

- **Basic idea:** Consider the lifting of a bilinear or quadratic inverse problem and apply the linear theory.
- **Main problem:** The solution has to be a rank-one tensor, which is a non-convex side condition.

Dilinear and diconvex mappings

Definition (Dilinear mapping)

A mapping $K\colon X \to Y$ is *dilinear* if there exists a linear mapping

$$\breve{K}\colon X \times (X \otimes_{\pi,\mathrm{sym}} X) \to Y \quad \text{such that} \quad K(u) = \breve{K}(u, u \otimes u)$$

for all u in X.

- A tensor w in $X \otimes_\pi X$ is symmetric if

$$w = \sum_{n=1}^{\infty} \lambda_n \, u_n \otimes v_n = \sum_{n=1}^{\infty} \lambda_n \, v_n \otimes u_n = w^{\mathsf{T}}.$$

- A dilinear mapping K is *bounded* if the representative $\breve{K}$ is bounded.

Lemma (Lifting of bounded dilinear mappings)

The representative $\breve{K}$ of a bounded dilinear mapping K is unique.

Example (Dilinear mappings)

The set of dilinear operators includes

- linear operators,
- quadratic operators,
- bilinear operators.

Definition (Diconvex mapping)

A mapping $F : X \to \overline{\mathbb{R}}$ is *diconvex* if there exists a proper, convex mapping

$$\breve{F}\colon X \times (X \otimes_{\pi,\mathrm{sym}} X) \to \overline{\mathbb{R}} \quad \text{such that} \quad F(u) = \breve{F}(u, u \otimes u)$$

for all u in X.

Definition (Dilinear subgradient)

Let $F \colon X \to \overline{\mathbb{R}}$ be a diconvex mapping. The dual element $(\xi, \Xi) \in X^* \times (X \otimes_{\pi,\mathrm{sym}} X)^*$ is a *dilinear subgradient* of F at u if $F(u)$ is finite and if

$$F(v) \geq F(u) + \langle \xi, v - u \rangle + \langle \Xi, v \otimes v - u \otimes u \rangle$$

for all v in X. The union of all dilinear subgradients of F at u is the *dilinear subdifferential* $\partial_\beta F(u)$. If no dilinear subgradient exists, the dilinear subdifferential is empty.

- **Figuratively:** Instead of the classical linear minorants

$$F(u) + \langle \xi, v - u \rangle$$

in convex analysis, we consider dilinear minorants

$$F(u) + \langle \xi, v - u \rangle + \langle \Xi, v \otimes v - u \otimes u \rangle \, .$$

M. Grasmair: *Generalized Bregman distances and convergence rates for non-convex regulatization methods.* (2010)

Definition (Representative subgradient)

The dual element $(\xi, \Xi) \in X^* \times (X \otimes_{\pi,\mathrm{sym}} X)^*$ is a *representative subgradient* of F at u with respect to $\check{F}$ if (ξ, Ξ) is a subgradient of the representative $\check{F}$ at $(u, u \otimes u)$. The union of all representative subgradient of F at u is the *representative subdifferential* $\check{\partial} F(u)$ with respect to $\check{F}$.

- Similar to the dilinear subgradient, the dual element (ξ, Ξ) is a representative subgradient if

$$\check{F}(v, w) \geq \check{F}(u, u \otimes u) + \langle \xi, v - u \rangle + \langle \Xi, w - u \otimes u \rangle$$

  for all (v, w) in $X \times (X \otimes_{\pi,\mathrm{sym}} X)$.
- For the dilinear subgradient, we only consider points of the form

$$(v, w) = (v, v \otimes v).$$

- The representative subgradient depends on the representative $\check{F}$.

- Let the mapping $F_\otimes$ be defined by

$$F_\otimes(u, w) := \begin{cases} F(u) & w = u \otimes u \\ +\infty & \text{else.} \end{cases}$$

- The convexification of $F_\otimes$ is given by

$$\operatorname{conv} F_\otimes(u, w) = \inf\left\{ \sum_{n=1}^{N} \alpha_n F(u_n) : (u, w) = \sum_{n=1}^{N} \alpha_n (u_n, u_n \otimes u_n) \right\}.$$

Theorem (Equality of subdifferentials)

Let $F\colon X \to \overline{\mathbb{R}}$ be a diconvex mapping. Then the representative subdifferental with respect to $\operatorname{conv} F_\otimes$ *and the dilinear subdifferential coincide, i.e.*

$$\breve{\partial} F(u) = \partial_\beta F(u).$$

- For the dilinear and representative subdifferential, we have the following computation rules:
  - sum rule for two diconvex mappings,
  - chain rule for linear operator and diconvex mapping,
  - chain rule for dilinear operator and convex mapping.

Definition (Dilinear BREGMAN distance)

Let $F\colon X \to \overline{R}$ be a diconvex mapping. The *dilinear BREGMAN domain* $\Delta_{\beta,\mathrm{dom}}(F)$ is defined by

$$\Delta_{\beta,\mathrm{dom}}(F) := \{u \in X : \partial_\beta F(u) \neq \emptyset\}.$$

For every $u \in \Delta_{\beta,\mathrm{dom}}(F)$ and $(\xi, \Xi) \in \partial_\beta F(u)$, the *dilinear distance* of v and u with respect to F and (ξ, Ξ) is given by

$$\Delta_{\beta,(\xi,\Xi)}(v, u) := F(v) - F(u) - \langle \xi, v - u\rangle - \langle \Xi, v \otimes v - u \otimes u\rangle \,.$$

Dilinear regularization

- For the dilinear inverse problem $K(u) = g^\dagger$, we consider the Tikhonov functional

$$J_\alpha(u) := \| K(u) - g^\delta \|^2 + \alpha\, R(u).$$

Assumption

Let X and Y be real Banach spaces with predual X_* and Y_*, where X_* is separable or reflexive. Assume that the data fidelity functional $S(\cdot) := \| K(\cdot) - g^\delta \|^2$ with the dilinear mapping $K\colon X \to Y$ and non-negative, proper, diconvex regularization functional $R\colon X \to \overline{\mathbb{R}}$ satisfy:

(i) *J_α is coercive in the sense that $J_u(u) \to +\infty$ whenever $\| u \| \to \infty$.*

(ii) *The functional R is sequentially weakly$*$ lower semi-continuous.*

(iii) *The dilinear operator K is sequentially weakly$*$ continuous.*

Proposition (Regularization properties)

The minimization of the Tikhonov functional J_α is

(i) *well-posed (existence of minimizers),*

(ii) *stable (convergence of minimizers),*

(iii) *consistent (convergence to a R-minimizing solution).*

Definition (R-minimizing solution)

A point $u^\dagger \in X$ is an *R-minimizing solution* if

$$K(u^\dagger) = g^\dagger \qquad \text{and} \qquad R(u^\dagger) \leq R(u)$$

for all further solutions u of $K(u) = g^\dagger$.

Theorem (Convergence rate)

Let Y be a Hilbert space, and let $u^\dagger$ be an R-minimizing solution of the dilinear inverse problem $K(u) = g^\dagger$. Under the made assumptions, and under the source condition

$$\breve{K}^* \omega = (\xi^\dagger, \Xi^\dagger) \in \partial_\beta R(u^\dagger)$$

for some ω in Y, the minimizer u_α^δ converges to $u^\dagger$ in the sense that

$$\Delta_{\beta,(\xi^\dagger,\Xi^\dagger)}(u_\alpha^\delta, u^\dagger) \le \left(\frac{\delta}{\sqrt{\alpha}} + \frac{\sqrt{\alpha}}{2}\|\omega\|\right)^2 \quad \text{and} \quad \|K(u_\alpha^\delta) - g^\delta\| \le \delta + \alpha\|\omega\|.$$

- **Motivation:** Generalizing Fermat's rule, we have

$$0 \in \partial_\beta(R + \chi_{K(u)=g^\dagger})(u^\dagger)$$

and, by applying the sum and chain rule,

$$\operatorname{ran} \breve{K}^* + \partial_\beta R(u^\dagger) \subset \partial_\beta(R + \chi_{K(u)=g^\dagger})(u^\dagger).$$

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Problem (Deautoconvolution)

Recover the unknown complex-valued signal $u \in L^2_{\mathbb{C}}([0,1])$ *from its kernel-based autoconvolution*

$$A_k[u](t) := \int_{-\infty}^{\infty} k(s,t)\, u(s)\, u(t-s)\, \mathrm{d}s \qquad (t \in \mathbb{R}).$$

- To apply the developed dilinear regularization theory, we consider $L^2_{\mathbb{C}}([0,1])$ and $L^2_{\mathbb{C}}([0,2])$ as real Hilbert spaces equipped with the inner product $\langle \cdot, \cdot \rangle_{\mathbb{R}} := \Re \langle \cdot, \cdot \rangle_{\mathbb{C}}$.

Problem (Tikhonov regularization)

Minimize the Tikhonov functional

$$J_\alpha(u) := \| \mathcal{A}_k[u] - g^\delta \|^2 + \alpha \| u \|^2.$$

- The Tikhonov functional J_α fulfils the required assumption.

Lemma (Adjoint autoconvolution)

The adjoint autoconvolution $\breve{\mathcal{A}}_k^*$ *maps a function* $\phi \in L^2_{\mathbb{C}}([0,2])$ *to the functional* $(0, \Phi) \in L^2_{\mathbb{C}}([0,1]) \times (L^2_{\mathbb{C}}([0,1]) \otimes_{\pi,\mathrm{sym}} L^2_{\mathbb{C}}([0,1]))^*$ *with*

$$\Phi[u](t) := \int_0^1 \overline{k(s, s+t)}\, \phi(s+t)\, \overline{u(s)}\, \mathrm{d}s \qquad (t \in [0,1]).$$

- Here Φ acts as a quadratic mapping by $u \mapsto \langle \Phi[u], u \rangle_{\mathbb{R}}$.

Theorem (Dilinear Subdifferential of Hilbert norms)

Let H *be a real Hilbert space with inner product* $\langle \cdot, \cdot \rangle_H$ *and norm* $\| \cdot \|_H$*. The dilinear subdifferential is given by*

$$\partial_\beta \| \cdot \|_H^2(u) = \{(-2Tu, \mathrm{Id} + T : T \in \mathcal{S}_-(H)\},$$

where $\mathcal{S}_-(H)$ *set of all self-adjoint and negative semi-definite operators on* H*.*

Theorem (Source condition)

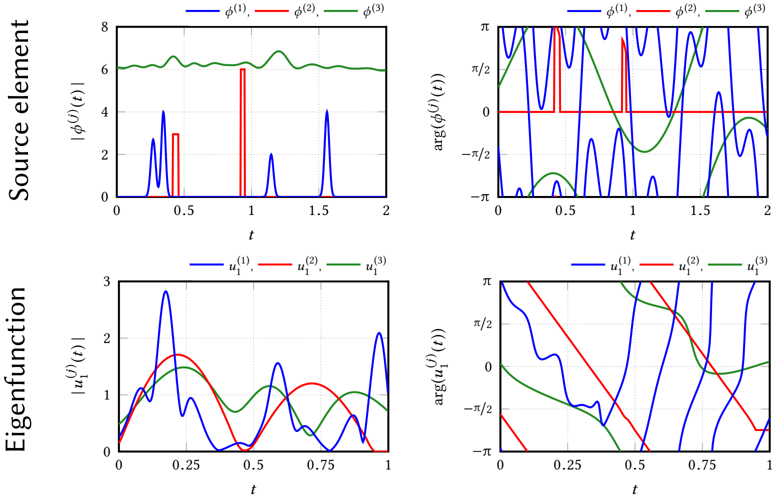
For a norm-minimizing solution $u^\dagger$ of the kernel-based deautoconvolution problem $\mathcal{A}_k[u] = g^\dagger$, the source condition $[\breve{K}^*\omega = (\xi^\dagger, \Xi^\dagger) \in \partial_\beta R(u^\dagger)]$ is fulfilled if and only if there exists a ϕ in $L^2_{\mathbb{C}}([0,2])$ such that

$$\|\Phi\| = 1 \qquad \text{and} \qquad \Phi[u^\dagger] = u^\dagger$$

for the integral operator

$$\Phi[u](t) := \int_0^1 \overline{k(s, s+t)}\, \phi(s+t)\, \overline{u(s)}\, \mathrm{d}s \qquad (t \in [0,1]).$$

Example (Norm-minimizing solutions)



Corollary (Convergence rate)

Let $u^\dagger$ be a norm-minimizing solution of the deautoconvolution problem $\mathcal{A}_k[u] = g^\dagger$. Under the previous source condition, the minimizer u_α^δ converges to $u^\dagger$ in the sense that

$$\Delta_{\beta,(0,\Phi)}(u_\alpha^\delta, u^\dagger) \leq \left(\frac{\delta}{\sqrt{\alpha}} + \frac{\sqrt{\alpha}}{2}\|\phi\|\right)^2$$

and

$$\|\mathcal{A}_k[u_\alpha^\delta] - g^\delta\| \leq \delta + \alpha\|\phi\|.$$

Theorem (Bregman distance)

Let Φ be an integral operator as before with distinct positive eigenvalues $1 = \lambda_1 > \lambda_2 > \dots$ and related finite-dimensional eigenspaces $E_1, E_2, \dots$ The corresponding dilinear Bregman distance is bounded from below by

$$\Delta_{\beta,(0,\Phi)}(v, u^\dagger) \geq (1 - \lambda_2)\|P_{E_1^\perp}(v - u^\dagger)\|^2.$$

Summary/Outlook

- Dilinear and diconvex mappings.
- Generalization of classical convex analysis.
- Regularization theory for dilinear inverse problems.
- Convergence rates for the deautoconvolution problem.

- Application of the dilinear theory to further problems.
- Algorithms to solve dilinear inverse problems numerically.

Thank you for the attention.

- R. Beinert, K. Bredies. *Regularization of bilinear and quadratic inverse problems by tensorial lifting.* (2018) – Preprint.