Sparse phase retrieval of one-dimensional signals by Prony's method

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Phase retrieval

Problem (Phase retrieval)

Recover the unknown sparse signal $f : \mathbb{R} \to \mathbb{C}$ of the form

$$f(t) := \sum_{j=1}^{N} c_j^{(0)}\, \delta(t - T_j) \qquad \text{or} \qquad f(t) := \sum_{j=1}^{N} c_j^{(m)}\, B_{j,m}(t)$$

with $c_j^{(m)} \in \mathbb{C}$, $T_j \in \mathbb{R}$ from its Fourier intensity

$$\left|\widehat{f}(\omega)\right| \qquad (\omega \in \mathbb{R}).$$

Definition (Fourier transform)

$$\widehat{f}(\omega) := \mathcal{F}[f](\omega) := \int_{-\infty}^{\infty} f(t)\, \mathrm{e}^{-\mathrm{i}\omega t}\, \mathrm{d}t \qquad (\omega \in \mathbb{R})$$

Example

Let f be a complex-valued signal. Then

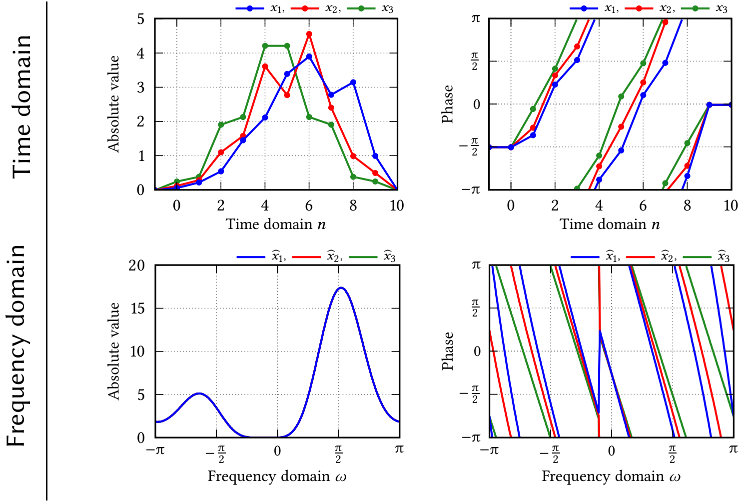
- the rotated signal $g := e^{i\alpha} f$ for $\alpha \in \mathbb{R}$,
- the shifted signal $g := f(\cdot - t_0)$ for $t_0 \in \mathbb{R}$,
- the conjugated, reflected signal $g := \overline{f(-\cdot)}$

have the same Fourier intensity $|\widehat{f}|$.

Definition (Trivial and non-trivial ambiguities)

A *trivial ambiguity* is caused by rotation, shift, or conjugated reflection. All other occurring ambiguities are called *non-trivial*.

Example (Non-trivial ambiguities)



Spike functions

- Let the unknown signal f be of the form

$$f(t) := \sum_{j=1}^{N} c_j^{(0)} \delta(t - T_j) \quad \text{with} \quad \widehat{f}(\omega) = \sum_{j=1}^{N} c_j^{(0)} \, \mathrm{e}^{-\mathrm{i}\omega T_j},$$

  where δ denotes the DIRAC delta function.
- The (distributional, squared) FOURIER intensity of f is given by

$$\left|\widehat{f}(\omega)\right|^2 = \sum_{j=1}^{N} \sum_{k=1}^{N} c_j^{(0)} \, \overline{c}_k^{(0)} \, \mathrm{e}^{-\mathrm{i}\omega(T_j - T_k)} \qquad (\omega \in \mathbb{R}).$$

- Assuming that the differences $T_j - T_k$ with $j \neq k$ are pairwise distinct, we can write the (squared) FOURIER intensity as exponential sum

$$P(\omega) := \left|\widehat{f}(\omega)\right|^2 = \sum_{\ell=-N(N-1)/2}^{N(N-1)/2} \gamma_\ell \, \mathrm{e}^{-\mathrm{i}\omega\tau_\ell}$$

  with $\tau_{-\ell} = -\tau_\ell$ and $\gamma_{-\ell} = \overline{\gamma}_\ell$.

- For $h > 0$ with $h\tau_\ell \in (-\pi, \pi)$, define the Prony polynomial Λ by

$$\Lambda(z) := \prod_{\ell=-N(N-1)/2}^{N(N-1)/2} \left(z - \mathrm{e}^{-\mathrm{i}h\tau_\ell}\right) = \sum_{k=0}^{N(N-1)+1} \lambda_k \, z^k,$$

  where $\lambda_{N(N-1)+1} = 1$.

- Using Λ, we have

$$\sum_{k=0}^{N(N-1)+1} \lambda_k \, P(h(k+m)) = \sum_{k=0}^{N(N-1)+1} \sum_{\ell=-N(N-1)/2}^{N(N-1)/2} \lambda_k \, \gamma_\ell \, \mathrm{e}^{-\mathrm{i}h(k+m)\tau_\ell}$$

$$= \sum_{\ell=-N(N-1)/2}^{N(N-1)/2} \gamma_\ell \, \mathrm{e}^{-\mathrm{i}hm\tau_\ell} \, \Lambda(\mathrm{e}^{-\mathrm{i}h\tau_\ell}) = 0$$

  for $m = 0, \ldots, N(N-1)$.

- The remaining coefficients $\lambda_0, \ldots, \lambda_{N(N-1)}$ can be determined by solving a system of linear equations.
- Compute the zeros $\mathrm{e}^{-\mathrm{i}h\tau_\ell}$ of Λ and the frequency differences τ_ℓ.
- Determine the coefficients γ_ℓ by solving the system

$$\sum_{\ell=-N(N-1)/2}^{N(N-1)/2} \gamma_\ell \, \mathrm{e}^{-\mathrm{i}hk\tau_\ell} = P(hk) \qquad (k = 0, \ldots, 2N(N-1)+1).$$

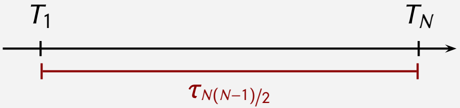
- Using the (conjugate) symmetries $\tau_{-\ell} = -\tau_\ell$ and $\gamma_{-\ell} = \overline{\gamma}_\ell$, the coefficients and frequency differences can be recovered form $P(kh)$ with $k = 0, \ldots, {}^3/_2\, N(N-1)$.

Recover the first and the last knot

- We want to recover the coefficients $c_j^{(0)}$ and knots T_j of the spike function with Fourier intensity

$$\left|\widehat{f}(\omega)\right|^2 = \sum_{j=1}^{N} \sum_{k=1}^{N} c_j^{(0)} \, \overline{c}_k^{(0)} \, \mathrm{e}^{-\mathrm{i}\omega(T_j - T_k)} = \sum_{\ell=-N(N-1)/2}^{N(N-1)/2} \gamma_\ell \, \mathrm{e}^{-\mathrm{i}\omega\tau_\ell}.$$

- Prony's method yields τ_ℓ (in increasing order) and γ_ℓ.
- Obviously, we have $\tau_{N(N-1)/2} = T_N - T_1$.
- Due to the shift ambiguity, set $T_1 = 0$ and $T_N = \tau_{N(N-1)/2}$.



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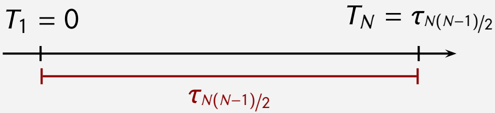
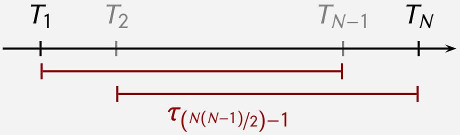


Recover the second knot

- Next, $\tau_{(N(N-1)/2)-1}$ corresponds either to $T_N - T_2$ or to $T_{N-1} - T_1$.
- Due to the conjugate reflection ambiguity, we assume $T_{N-1} = \tau_{(N(N-1)/2)-1}$.
- There is an index ℓ^* such that $\tau_{\ell^*} + \tau_{(N(N-1)/2)-1} = \tau_{N(N-1)/2}$.
- The corresponding coefficients are

$$\gamma_{N(N-1)/2} = c_N^{(0)}\,\overline{c}_1^{(0)}, \quad \gamma_{(N(N-1)/2)-1} = c_{N-1}^{(0)}\,\overline{c}_1^{(0)} \quad \text{and} \quad \gamma_{\ell^*} = c_N^{(0)}\,\overline{c}_{N-1}^{(0)}.$$



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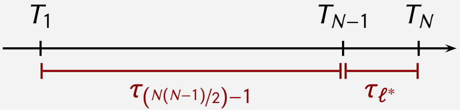


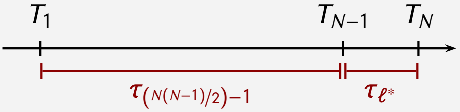
- Solving this equation system, we obtain

$$\left|c_1^{(0)}\right|^2 = \frac{\gamma_{N(N-1)/2}\,\overline{\gamma}_{(N(N-1)/2)-1}}{\gamma_{\ell^*}}$$

and

$$c_N^{(0)} = \frac{\gamma_{N(N-1)/2}}{\overline{c}_1^{(0)}}, \qquad c_{N-1}^{(0)} = \frac{\gamma_{(N(N-1)/2)-1}}{\overline{c}_1^{(0)}}.$$

- Due to the rotation ambiguity, we can assume $c_1^{(0)}$ is real and non-negative.



- The distance $\tau_{(N(N-1)/2)-2}$ is related to $T_N - T_2$ or $T_{N-2} - T_1$.
- Again, there is an ℓ^* so that $\tau_{\ell^*} + \tau_{(N(N-1)/2)-2} = \tau_{N(N-1)/2}$.

<u>Case 1</u>: $\tau_{(N(N-1)/2)-2} = T_N - T_2$

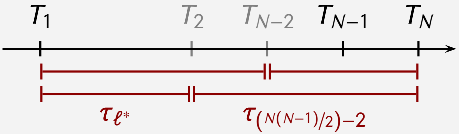
$$c_2^{(0)} = \frac{\gamma_{\ell^*}}{\overline{c}_1^{(0)}} = \frac{\overline{\gamma}_{(N(N-1)/2)-2}}{\overline{c}_N^{(0)}}$$

<u>Case 2</u>: $\tau_{(N(N-1)/2)-2} = T_{N-2} - T_1$

$$c_{N-2}^{(0)} = \frac{\overline{\gamma}_{\ell^*}}{\overline{c}_N^{(0)}} = \frac{\gamma_{(N(N-1)/2)-2}}{\overline{c}_1^{(0)}}$$

- If the equations in both cases are true, then we have

$$\left|\frac{c_N^{(0)}}{c_1^{(0)}}\right| = \left|\frac{\gamma_{(N(N-1)/2)-2}}{\gamma_{\ell^*}}\right| = \left|\frac{c_1^{(0)}}{c_N^{(0)}}\right| \quad \text{and thus} \quad \left|c_1^{(0)}\right| = \left|c_N^{(0)}\right|.$$



- The distance $\tau_{(N(N-1)/2)-2}$ is related to $T_N - T_2$ or $T_{N-2} - T_1$.
- Again, there is an ℓ^* so that $\tau_{\ell^*} + \tau_{(N(N-1)/2)-2} = \tau_{N(N-1)/2}$.

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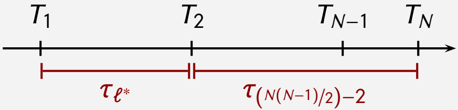


Theorem (BEINERT, PLONKA [2017])

Let f be a spike function. If

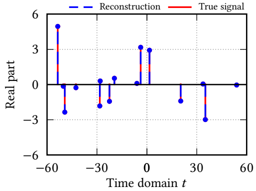
- *the knot differences $T_j - T_k$ differ pairwise for $j, k \in \{1, \ldots, N\}$, $j \neq k$*
- *the coefficients satisfy $|c_1^{(0)}| \neq |c_N^{(0)}|$*
- *the step size $h > 0$ fulfils $h(T_j - T_k) \in (-\pi, \pi)$ for all j, k*

then f can be uniquely recovered from its FOURIER intensities $|\mathcal{F}[f](h\ell)|$ with $\ell = 0, \ldots, 3/2\, N(N-1)$ up to trivial ambiguities.

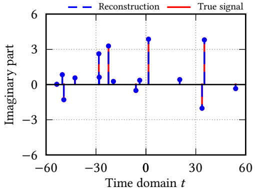
Corollary

Almost all spike functions f can be uniquely recovered from their Fourier intensities $|\mathcal{F}[f]|$ up to trivial ambiguities.

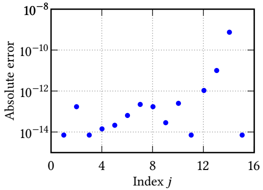
Example



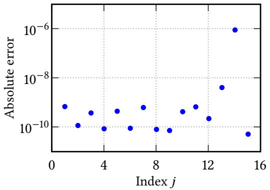
Real part: $\Re f(t)$



Imaginary part: $\Im f(t)$



Absolute error of the knots



Absolute error of the coefficients

Recovery of f from $|\mathcal{F}[f](h\ell)|$ with $\ell = 0, \ldots, 1000$.

Spline functions

- For $m \geq 1$ and $T_1 < \cdots < T_{N+m}$, let the unknown signal f be a spline function of order m, i.e.

$$f(t) := \sum_{j=1}^{N} c_j^{(m)} B_{j,m}(t) \qquad (t \in \mathbb{R}),$$

where

$$B_{j,m}(t) := \tfrac{t-T_j}{T_{j+m-1}-T_j} B_{j,m-1}(t) + \tfrac{T_{j+m}-t}{T_{j+m}-T_{j+1}} B_{j+1,m-1}(t)$$

and

$$B_{j,1}(t) := \mathbb{1}_{[T_j,T_{j+1})}(t) := \begin{cases} 1 & t \in [T_j, T_{j+1}), \\ 0 & \text{else.} \end{cases}$$

- For $0 \leq k \leq m-2$, the kth derivative of f is given by

$$\frac{\mathrm{d}^k}{\mathrm{d}t^k} f(t) = \sum_{j=1}^{N+k} c_j^{(m-k)} \, B_{j,m-k}(t)$$

with

$$c_j^{(m-k)} := (m-k) \, \frac{c_j^{(m-k+1)} - c_{j-1}^{(m-k+1)}}{T_{j+m-k} - T_j}.$$

- For $k = m-1$, we obtain a step function, which corresponds to the right derivative of the linear spline $f^{(m-2)}$.
- In a distributional manner, the mth derivative is the spike function

$$\frac{\mathrm{d}^m}{\mathrm{d}t^m} f(t) = \sum_{j=1}^{N+m} c_j^{(0)} \, \delta(t - T_j)$$

with

$$c_j^{(0)} := c_j^{(1)} - c_{j-1}^{(1)}.$$

- For the Fourier transform of the mth derivative, we obtain

$$\widehat{f^{(m)}}(\omega) = (\mathrm{i}\omega)^m \hat{f}(\omega) = \sum_{j=1}^{N+m} c_j^{(0)} \, \mathrm{e}^{-\mathrm{i}\omega T_j}$$

and thus

$$\omega^{2m} \left|\hat{f}(\omega)\right|^2 = \sum_{j=1}^{N+m} \sum_{k=1}^{N+m} c_j^{(0)} \, \overline{c}_k^{(0)} \, \mathrm{e}^{-\mathrm{i}\omega(T_j - T_k)}.$$

- In order to determine T_j and $c_j^{(0)}$, we consider the exponential sum

$$P(\omega) := \omega^{2m} \left|\hat{f}(\omega)\right|^2 = \sum_{\ell=-(N+m)(N+m-1)/2}^{(N+m)(N+m-1)/2} \gamma_\ell \, \mathrm{e}^{-\mathrm{i}\omega\tau_\ell}$$

and apply the proposed method.

Theorem (Beinert, Plonka [2017])

Let f be a spline function of order m. If

- *the knot differences $T_j - T_k$ differ pairwise for $j, k \in \{1, \dots, N\}$, $j \neq k$*
- *the coefficients satisfy $|c_1^{(0)}| \neq |c_{N+m}^{(0)}|$*
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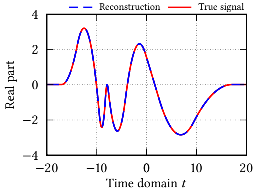
then f can be uniquely recovered from its Fourier intensities $|\mathcal{F}[f](h\ell)|$ with $\ell = 0, \dots, 3/2\,(N+m)(N+m-1)$ up to trivial ambiguities.

Corollary

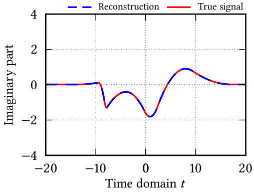
Almost all spline functions f of order m can be uniquely recovered from their Fourier intensities $|\mathcal{F}[f]|$ up to trivial ambiguities.

Numerical example (spline function)

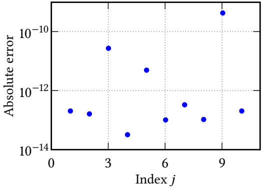
Example



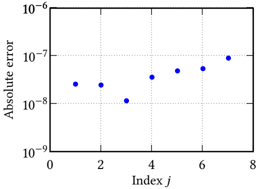
Real part: $\Re f(t)$



Imaginary part: $\Im f(t)$



Absolute error of the knots



Absolute error of the coefficients

Recovery of f from $|\mathcal{F}[f](h\ell)|$ with $\ell = 0, \ldots, 400$.

Summary/Outlook

- Phase retrieval for spike and spline functions.
- Uniqueness for almost every signal.
- Numerical method to recover signals without noise.

- Adapt the algorithm for disturbed measurements.
- Approximate a given Fourier intensity by a spline in the time domain.

Thank you for the attention.

- R. Beinert, G. Plonka. Sparse phase retrieval of one-dimensional signals by Prony's method. *Frontiers in Applied Mathematics and Statistics.* **3**(5), 2017.