

Phase retrieval of sparse continuous-time signals by PRONY's method

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*SIAM Conference on Applied Algebraic Geometry
(Bern, Switzerland)*

10 July 2019



PHASE RETRIEVAL

PROBLEM FORMULATION

Problem (Phase retrieval)

Recover the *unknown sparse signal* $f: \mathbb{R} \rightarrow \mathbb{C}$ of the form

$$f(t) := \sum_{j=1}^N c_j^{(0)} \delta(t - T_j) \quad \text{or} \quad f(t) := \sum_{j=1}^N c_j^{(m)} B_{j,m}(t)$$

with $c_j^{(m)} \in \mathbb{C}$, $T_j \in \mathbb{R}$ from its *FOURIER intensity*

$$|\widehat{f}(\omega)| \quad (\omega \in \mathbb{R}).$$

Definition (FOURIER transform)

$$\widehat{f}(\omega) := \mathcal{F}[f](\omega) := \int_{-\infty}^{\infty} f(t) e^{-i\omega t} dt \quad (\omega \in \mathbb{R})$$

TRIVIAL AND NON-TRIVIAL AMBIGUITIES

Example

Let f be a complex-valued signal. Then

- the **rotated** signal $g := e^{i\alpha} f$ for $\alpha \in \mathbb{R}$,
- the **shifted** signal $g := f(\cdot - t_0)$ for $t_0 \in \mathbb{R}$,
- the **conjugated, reflected** signal $g := \overline{f(-\cdot)}$

have the same FOURIER intensity $|\widehat{f}|$.

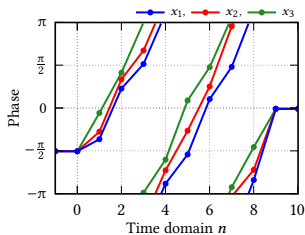
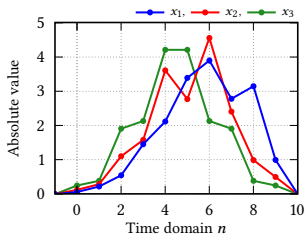
Definition (Trivial and non-trivial ambiguities)

A **trivial ambiguity** is caused by **rotation**, **shift**, or **conjugated reflection**. All other occurring ambiguities are called *non-trivial*.

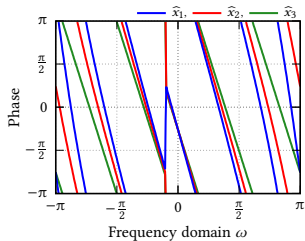
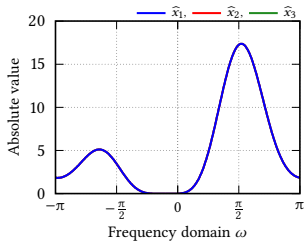
NON-TRIVIAL AMBIGUITIES IN THE DISCRETE SETTING

Example (Non-trivial ambiguities)

Time domain



Frequency domain



SPIKE FUNCTIONS

PHASE RETRIEVAL FOR SPIKE FUNCTIONS

- Let the unknown signal f be of the form

$$f(t) := \sum_{j=1}^N c_j^{(0)} \delta(t - T_j) \quad \text{with} \quad \widehat{f}(\omega) = \sum_{j=1}^N c_j^{(0)} e^{-i\omega T_j},$$

where δ denotes the DIRAC delta function.

- The (distributional, squared) FOURIER intensity of f is given by

$$|\widehat{f}(\omega)|^2 = \sum_{j=1}^N \sum_{k=1}^N c_j^{(0)} \overline{c_k^{(0)}} e^{-i\omega(T_j - T_k)} \quad (\omega \in \mathbb{R}).$$

- Assuming that the differences $T_j - T_k$ with $j \neq k$ are pairwise distinct, we can write the (squared) FOURIER intensity as exponential sum

$$P(\omega) := |\widehat{f}(\omega)|^2 = \sum_{\ell=-N(N-1)/2}^{N(N-1)/2} \gamma_\ell e^{-i\omega \tau_\ell}$$

with $\tau_{-\ell} = -\tau_\ell$ and $\gamma_{-\ell} = \overline{\gamma}_\ell$.

- For $h > 0$ with $h\tau_\ell \in (-\pi, \pi)$, define the **PRONY polynomial** Λ by

$$\Lambda(z) := \prod_{\ell=-N(N-1)/2}^{N(N-1)/2} (z - e^{-ih\tau_\ell}) = \sum_{k=0}^{N(N-1)+1} \lambda_k z^k,$$

where $\lambda_{N(N-1)+1} = 1$.

- Using Λ , we have

$$\begin{aligned} \sum_{k=0}^{N(N-1)+1} \lambda_k P(h(k+m)) &= \sum_{k=0}^{N(N-1)+1} \sum_{\ell=-N(N-1)/2}^{N(N-1)/2} \lambda_k \gamma_\ell e^{-ih(k+m)\tau_\ell} \\ &= \sum_{\ell=-N(N-1)/2}^{N(N-1)/2} \gamma_\ell e^{-ihm\tau_\ell} \Lambda(e^{-ih\tau_\ell}) = 0 \end{aligned}$$

for $m = 0, \dots, N(N-1)$.

- The remaining coefficients $\lambda_0, \dots, \lambda_{N(N-1)}$ can be determined by solving a system of linear equations.
- Compute the zeros $e^{-ih\tau_\ell}$ of Λ and the **frequency differences** τ_ℓ .
- Determine the **coefficients** γ_ℓ by solving the system

$$\sum_{\ell=-N(N-1)/2}^{N(N-1)/2} \gamma_\ell e^{-ihk\tau_\ell} = P(hk) \quad (k = 0, \dots, 2N(N-1) + 1).$$

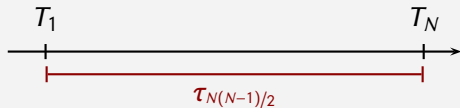
- Using the (conjugate) symmetries $\tau_{-\ell} = -\tau_\ell$ and $\gamma_{-\ell} = \bar{\gamma}_\ell$, the coefficients and frequency differences can be recovered from $P(kh)$ with $k = 0, \dots, 3/2 N(N-1)$.

RECOVER THE FIRST AND THE LAST KNOT

- We want to recover the coefficients $c_j^{(0)}$ and knots T_j of the spike function with FOURIER intensity

$$|\widehat{f}(\omega)|^2 = \sum_{j=1}^N \sum_{k=1}^N c_j^{(0)} \overline{c_k^{(0)}} e^{-i\omega(T_j - T_k)} = \sum_{\ell=-N(N-1)/2}^{N(N-1)/2} \gamma_\ell e^{-i\omega\tau_\ell}.$$

- PRONY's method yields τ_ℓ (in increasing order) and γ_ℓ .
- Obviously, we have $\tau_{N(N-1)/2} = T_N - T_1$.
- Due to the shift ambiguity, set $T_1 = 0$ and $T_N = \tau_{N(N-1)/2}$.

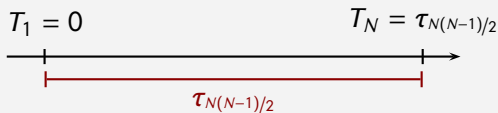


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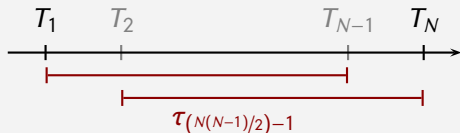
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RECOVER THE SECOND KNOT

- Next, $\tau_{(N(N-1)/2)-1}$ corresponds either to $T_N - T_2$ or to $T_{N-1} - T_1$.
- Due to the conjugate reflection ambiguity, we assume
$$T_{N-1} = \tau_{(N(N-1)/2)-1}.$$
- There is an index ℓ^* such that $\tau_{\ell^*} + \tau_{(N(N-1)/2)-1} = \tau_{N(N-1)/2}$.
- The corresponding coefficients are

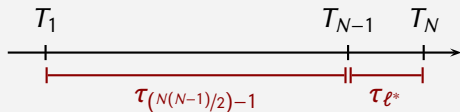
$$\gamma_{N(N-1)/2} = c_N^{(0)} \overline{c_1^{(0)}}, \quad \gamma_{(N(N-1)/2)-1} = c_{N-1}^{(0)} \overline{c_1^{(0)}} \quad \text{and} \quad \gamma_{\ell^*} = c_N^{(0)} \overline{c_{N-1}^{(0)}}.$$



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$$\gamma_{N(N-1)/2} = c_N^{(0)} \bar{c}_1^{(0)}, \quad \gamma_{(N(N-1)/2)-1} = c_{N-1}^{(0)} \bar{c}_1^{(0)} \quad \text{and} \quad \gamma_{\ell^*} = c_N^{(0)} \bar{c}_{N-1}^{(0)}.$$



RECOVER THE RELATED COEFFICIENTS

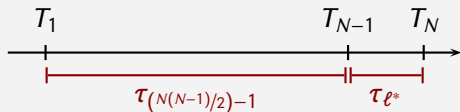
- Solving this equation system, we obtain

$$\left| c_1^{(0)} \right|^2 = \frac{\gamma_{N(N-1)/2} \bar{\gamma}_{(N(N-1)/2)-1}}{\gamma_{\ell^*}}$$

and

$$c_N^{(0)} = \frac{\gamma_{N(N-1)/2}}{c_1^{(0)}}, \quad c_{N-1}^{(0)} = \frac{\gamma_{(N(N-1)/2)-1}}{c_1^{(0)}}.$$

- Due to the rotation ambiguity, we can assume $c_1^{(0)}$ is real and non-negative.



ITERATIVELY RECOVER THE REMAINING KNOTS AND COEFFICIENTS

- The distance $\tau_{(N(N-1)/2)-2}$ is related to $T_N - T_2$ or $T_{N-2} - T_1$.
- Again, there is an ℓ^* so that $\tau_{\ell^*} + \tau_{(N(N-1)/2)-2} = \tau_{N(N-1)/2}$.

Case 1: $\tau_{(N(N-1)/2)-2} = T_N - T_2$

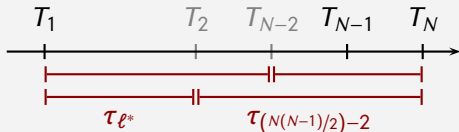
$$c_2^{(0)} = \frac{\gamma_{\ell^*}}{c_1^{(0)}} = \frac{\bar{\gamma}_{(N(N-1)/2)-2}}{c_N^{(0)}}$$

Case 2: $\tau_{(N(N-1)/2)-2} = T_{N-2} - T_1$

$$c_{N-2}^{(0)} = \frac{\bar{\gamma}_{\ell^*}}{c_N^{(0)}} = \frac{\gamma_{(N(N-1)/2)-2}}{c_1^{(0)}}$$

- If the equations in both cases are true, then we have

$$\left| \frac{c_N^{(0)}}{c_1^{(0)}} \right| = \left| \frac{\gamma_{(N(N-1)/2)-2}}{\gamma_{\ell^*}} \right| = \left| \frac{c_1^{(0)}}{c_N^{(0)}} \right| \quad \text{and thus} \quad \left| c_1^{(0)} \right| = \left| c_N^{(0)} \right|.$$



ITERATIVELY RECOVER THE REMAINING KNOTS AND COEFFICIENTS

- The distance $\tau_{(N(N-1)/2)-2}$ is related to $T_N - T_2$ or $T_{N-2} - T_1$.
- Again, there is an ℓ^* so that $\tau_{\ell^*} + \tau_{(N(N-1)/2)-2} = \tau_{N(N-1)/2}$.

Case 1: $\tau_{(N(N-1)/2)-2} = T_N - T_2$

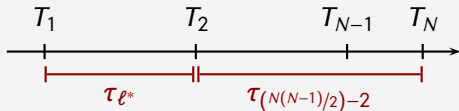
$$c_2^{(0)} = \frac{\gamma_{\ell^*}}{c_1^{(0)}} = \frac{\bar{\gamma}_{(N(N-1)/2)-2}}{c_N^{(0)}}$$

Case 2: $\tau_{(N(N-1)/2)-2} = T_{N-2} - T_1$

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- If the equations in both cases are true, then we have

$$\left| \frac{c_N^{(0)}}{c_1^{(0)}} \right| = \left| \frac{\gamma_{(N(N-1)/2)-2}}{\gamma_{\ell^*}} \right| = \left| \frac{c_1^{(0)}}{c_N^{(0)}} \right| \quad \text{and thus} \quad \left| c_1^{(0)} \right| = \left| c_N^{(0)} \right|.$$



Theorem (BEINERT, PLONKA [2017])

Let f be a spike function. If

- the knot differences $T_j - T_k$ differ pairwise for $j, k \in \{1, \dots, N\}, j \neq k$
- the coefficients satisfy $|c_1^{(0)}| \neq |c_N^{(0)}|$
- the step size $h > 0$ fulfils $h(T_j - T_k) \in (-\pi, \pi)$ for all j, k

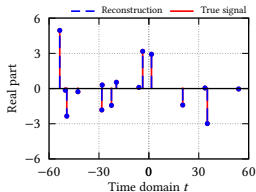
then f can be uniquely recovered from its FOURIER intensities $|\mathcal{F}[f](h\ell)|$ with $\ell = 0, \dots, 3/2 N(N - 1)$ up to trivial ambiguities.

Corollary

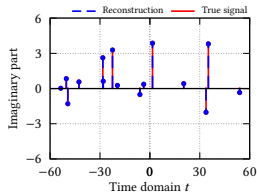
Almost all spike functions f can be uniquely recovered from their Fourier intensities $|\mathcal{F}[f]|$ up to trivial ambiguities.

NUMERICAL EXAMPLE (SPIKE FUNCTION)

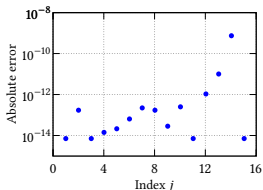
Example



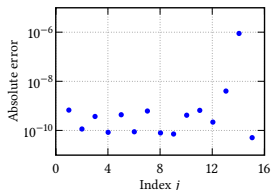
Real part: $\Re f(t)$



Imaginary part: $\Im f(t)$



Absolute error of the knots



Absolute error of the coefficients

Recovery of f from $|\mathcal{F}[f](h\ell)|$ with $\ell = 0, \dots, 1000$.

SPLINE FUNCTIONS

- For $m \geq 1$ and $T_1 < \dots < T_{N+m}$, let the unknown signal f be a spline function of order m , i.e.

$$f(t) := \sum_{j=1}^N c_j^{(m)} B_{j,m}(t) \quad (t \in \mathbb{R}),$$

where

$$B_{j,m}(t) := \frac{t-T_j}{T_{j+m-1}-T_j} B_{j,m-1}(t) + \frac{T_{j+m}-t}{T_{j+m}-T_{j+1}} B_{j+1,m-1}(t)$$

and

$$B_{j,1}(t) := \mathbb{1}_{[T_j, T_{j+1})}(t) := \begin{cases} 1 & t \in [T_j, T_{j+1}), \\ 0 & \text{else.} \end{cases}$$

- For $0 \leq k \leq m - 2$, the k th derivative of f is given by

$$\frac{d^k}{dt^k} f(t) = \sum_{j=1}^{N+k} c_j^{(m-k)} B_{j,m-k}(t)$$

with

$$c_j^{(m-k)} := (m-k) \frac{c_j^{(m-k+1)} - c_{j-1}^{(m-k+1)}}{T_{j+m-k} - T_j}.$$

- For $k = m - 1$, we obtain a **step function**, which corresponds to the right derivative of the linear spline $f^{(m-2)}$.
- In a distributional manner, the m th derivative is the **spike function**

$$\frac{d^m}{dt^m} f(t) = \sum_{j=1}^{N+m} c_j^{(0)} \delta(t - T_j)$$

with

$$c_j^{(0)} := c_j^{(1)} - c_{j-1}^{(1)}.$$

- For the FOURIER transform of the m th derivative, we obtain

$$\widehat{f^{(m)}}(\omega) = (i\omega)^m \widehat{f}(\omega) = \sum_{j=1}^{N+m} c_j^{(0)} e^{-i\omega T_j}$$

and thus

$$\omega^{2m} |\widehat{f}(\omega)|^2 = \sum_{j=1}^{N+m} \sum_{k=1}^{N+m} c_j^{(0)} \overline{c_k^{(0)}} e^{-i\omega(T_j - T_k)}.$$

- In order to determine T_j and $c_j^{(0)}$, we consider the exponential sum

$$P(\omega) := \omega^{2m} |\widehat{f}(\omega)|^2 = \sum_{\ell=-(N+m)(N+m-1)/2}^{(N+m)(N+m-1)/2} \gamma_\ell e^{-i\omega\tau_\ell}$$

and apply the proposed method.

UNIQUENESS FOR SPLINE FUNCTIONS

Theorem (BEINERT, PLONKA [2017])

Let f be a spline function of order m . If

- the knot differences $T_j - T_k$ differ pairwise for $j, k \in \{1, \dots, N\}, j \neq k$*
- the coefficients satisfy $|c_1^{(0)}| \neq |c_{N+m}^{(0)}|$*
- the step size $h > 0$ fulfils $h(T_j - T_k) \in (-\pi, \pi)$ for all j, k*

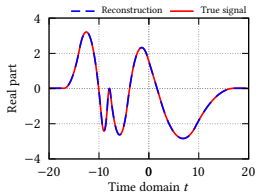
then f can be uniquely recovered from its FOURIER intensities $|\mathcal{F}[f](h\ell)|$ with $\ell = 0, \dots, 3/2(N+m)(N+m-1)$ up to trivial ambiguities.

Corollary

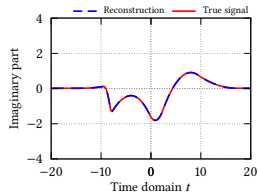
Almost all spline functions f of order m can be uniquely recovered from their Fourier intensities $|\mathcal{F}[f]|$ up to trivial ambiguities.

NUMERICAL EXAMPLE (SPLINE FUNCTION)

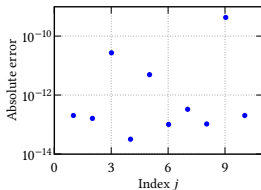
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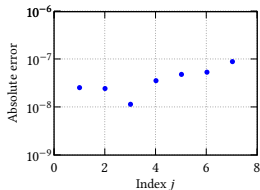
Real part: $\Re f(t)$



Imaginary part: $\Im f(t)$



Absolute error of the knots



Absolute error of the coefficients

Recovery of f from $|\mathcal{F}[f](h\ell)|$ with $\ell = 0, \dots, 400$.

- Phase retrieval for spike and spline functions.
 - Uniqueness for almost every signal.
 - Numerical method to recover signals without noise.
-
- Adapt the algorithm for disturbed measurements.
 - Approximate a given Fourier intensity by a spline in the time domain.

Thank you for the attention.

- R. BEINERT, G. PLONKA. Sparse phase retrieval of one-dimensional signals by Prony's method. *Frontiers in Applied Mathematics and Statistics*. 3(5), 2017.