Phase Retrieval and System Identification in Dynamical Sampling via Prony's Method

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How to Model the Spatiotemporal Samples Mathematically?

- Consider the discrete setting.
- Unknown signal $\boldsymbol{x} \in \mathbb{C}^d$.
- System matrix $\boldsymbol{A}^\top \in \mathbb{C}^{d \times d}$ for one time step.
- Known sensors $\langle \cdot, \boldsymbol{\phi}_i \rangle$ with $\boldsymbol{\phi}_i \in \mathbb{C}^d$.

Discrete Recovery Problem

- Recover $\boldsymbol{x}$ from the measurements

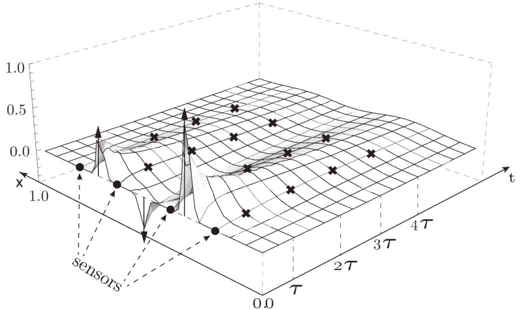
$$\left\{ \langle (\boldsymbol{A}^\top)^\ell \boldsymbol{x}, \boldsymbol{\phi}_i \rangle \right\}_{\ell,i=0}^{L-1,J-1} = \left\{ \langle \boldsymbol{x}, \boldsymbol{A}^\ell \boldsymbol{\phi}_i \rangle \right\}_{\ell,i=0}^{L-1,J-1}$$

using J sensors and L time steps.

Applications

- Repeated convolutions, Gabor systems, Frenel diffraction.

Dynamical Sampling



- Spatiotemporal samples of a diffusion field.

[Ranieri, Chebira, Lu & Vetterli 2011]

Recovery Guarantees

1. When does $\boldsymbol{A}$ and $\boldsymbol{\phi}_i$ allow the recovery of $\boldsymbol{x}$?
2. When does $\boldsymbol{A}^\ell \boldsymbol{\phi}_i$ form a frame?

[Lu & Vetterli 2009], [Aldroubi, Cabrelli, Molter & Tang 2007], [Aldroubi, Davis & Krishtal 2013]

System Identification

- Can we also recover $\boldsymbol{A}$?
  1. Conditions to identify the spectrum of $\boldsymbol{A}$.
  2. Recovery process if $\boldsymbol{A}$ is a circulant matrix.

[Aldroubi & Krishtal 2016], [Tang 2017]

Phase Retrieval

- Recover $\boldsymbol{x}$ and maybe partially $\boldsymbol{A}$ from the measurements

$$\left|\langle \boldsymbol{x}, \boldsymbol{A}^\ell \boldsymbol{\phi}_i \rangle\right|, \qquad \ell = 0, \dots, L-1, i = 0, \dots, J-1.$$

1. Recovery of $\boldsymbol{x} \in \mathbb{R}^d$ may be ensured by the complement property. [Aldroubi, Krishtal & Tang 2020]
2. Recovery of $\boldsymbol{x} \in \mathbb{C}^d$ using polarization techniques. [RB & MH 2021]

- **Problem:** Recover $\eta_k \in \mathbb{C}_*$ and $\beta_k \in \mathbb{C}_*$ of the exponential sum

$$f(t) := \sum_{k=0}^{K-1} \eta_k \, \beta_k^t \qquad \text{from} \qquad h_\ell := f(\ell), \quad \ell = 0, \ldots, 2K-1.$$

- **Prony Polynomial:** Define $P(z) := \sum_{k=0}^{K} \gamma_k z^k = \prod_{k=0}^{K-1} (z - \beta_k)$ with $\gamma_K = 1$.
- **Key Observation:** The samples satisfy

$$\sum_{k=0}^{K} \gamma_k \, h_{\ell+k} = \sum_{k=0}^{K} \sum_{j=0}^{K-1} \gamma_k \, \eta_j \, \beta_j^{k+\ell} = \sum_{j=0}^{K-1} \eta_j \beta_j^\ell \, P(\beta_j) = 0, \qquad \ell = 0, \ldots, K-1.$$

- **Prony's method:**
  1. Calculate γ_k by solving a linear equation system.
  2. Extract β_k via the roots of P.
  3. Determine η_k by solving a linear equation system.

- Let $A = S\Lambda S^{-1}$ be a diagonalizable with distinct eigenvalues.
- With $y := S^* x$ and $\psi := S^{-1}\phi$, the measurements are

$$|\langle x, A^\ell \phi\rangle|^2 = |\langle x, S\Lambda^\ell S^{-1}\phi\rangle|^2 = |\langle y, \Lambda^\ell \psi\rangle|^2 = \left|\sum_{k=0}^{d-1} \lambda_k^\ell \underbrace{\bar{y}_k \psi_k}_{=:c_k}\right|^2 = \sum_{j,k=0}^{d-1} c_j \bar{c}_k (\lambda_j \bar{\lambda}_k)^\ell = \sum_{m=0}^{d^2-1} \eta_m \beta_m^\ell .$$

- **Key Problem:** Which m corresponds to which j and k.

Definition (Spectral Persistence)

Let $A = S\Lambda S^{-1}$ be a diagonalizable with distinct eigenvalues. We say that a given vector $\phi \in \mathbb{C}^d$ is *A-spectrally persistent* if $S^{-1}\phi$ does not vanish anywhere.

Definition (Collision Freedom)

A set $M := \{\mu_0, \dots, \mu_{d-1}\} \subset \mathbb{C}$ is called

- *collision-free* if the products $\mu_j \bar{\mu}_k$ are pairwise distinct for $j, k \in \{0, \dots, d-1\}$.
- *absolutely collision-free* if M is collision-free and if the products $|\mu_j||\mu_k|$ are pairwise distinct for $j > k$.

Exclusive Phase Retrieval or System Identification

- The measurement are an exponential sum of order d^2 given by

$$|\langle \mathbf{x}, A^\ell \boldsymbol{\phi}\rangle|^2 = \sum_{j,k=0}^{d-1} c_j \bar{c}_k (\lambda_j \bar{\lambda}_k)^\ell = \sum_{m=0}^{d^2-1} \eta_m \beta_m^\ell \qquad (c_k := \bar{y}_k \psi_k,\ \mathbf{y} := \mathbf{S}^* \mathbf{x},\ \boldsymbol{\psi} := \mathbf{S}^{-1} \boldsymbol{\phi}).$$

Theorem (Exclusive Phase Retrieval)

Let $A \in \mathbb{C}^{d\times d}$ be known and diagonalizable with collision-free eigenvalues, and let $\boldsymbol{\phi} \in \mathbb{C}^d$ be A-spectrally persistent. Then every $\mathbf{x} \in \mathbb{C}^d$ can be recovered from the samples $\{|\langle \mathbf{x}, A^\ell \boldsymbol{\phi}\rangle|\}_{\ell=0}^{d^2-1}$ up to global phase.

Theorem (Exclusive System Identification)

Let $A = \mathbf{S}\Lambda\mathbf{S}^{-1}$ be diagonalizable by a known eigenvector basis $\mathbf{S}$ and assume that the eigenvalues are collision-free. Let $\boldsymbol{\phi} \in \mathbb{C}^d$ be A-spectrally persistent, and let $\mathbf{x} \in \mathbb{C}^d$ be given. If the coefficients c_k are collision-free too, then the eigenvalues $\lambda_0, \ldots, \lambda_{d-1}$ of A are defined by the samples $\{|\langle \mathbf{x}, A^\ell \boldsymbol{\phi}\rangle|\}_{\ell=0}^{2d^2-1}$ up to global phase.

Theorem (Spectrum Identification)

Let $A = S\Lambda S^{-1}$ be diagonalizable by a known eigenvector basis S and assume that the eigenvalues are absolutely collision-free. Let $\phi \in \mathbb{C}^d$ be A-spectrally persistent, and let $y := S^* x$ be elementwise non-zero for unknown $x \in \mathbb{C}^d$. Then the spectrum of A is determined by the samples $\{|\langle x, A^\ell \phi\rangle|\}_{\ell=0}^{2d^2-1}$ up to global phase and winding direction.

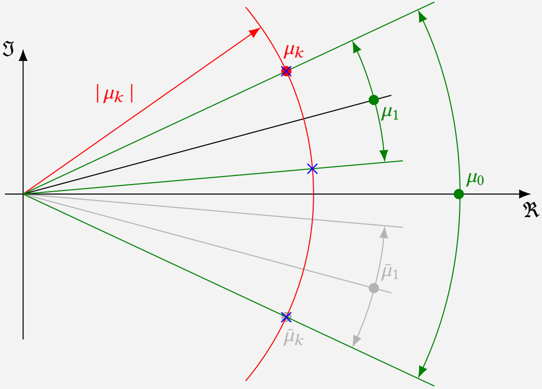
Definition (Winding Direction)

For a given vector $x = (x_0, \ldots, x_{d-1})$, we call the set of relative phases $\arg(x_j \bar{x}_k)$ the *winding direction* of x.

- Denote the recovered eigenvalues of A by μ_k with $|\mu_0| > \cdots > |\mu_{d-1}|$.
- The collision freedom guarantees $\mu_j \bar{\mu}_k \neq \mu_k \bar{\mu}_j$, i.e. $\Im[\mu_j \bar{\mu}_k] = -\Im[\mu_k \bar{\mu}_j] \neq 0$, for $j \neq k$.
- Real bases β_k correspond to the magnitudes $|\mu_k|$.
- The absolute collision freedom gives $\mu_j \bar{\mu}_k$ and $\mu_k \bar{\mu}_j$ corresponding to $|\mu_j|$ and $|\mu_k|$.

Spectrum Identification

- Recover the phases:



Problem

- We only get the spectrum but not the order of the eigenvalues.

Convolution Operators

- All circulant matrices are diagonalizable as

$$\operatorname{circ}(\boldsymbol{a}) = {}^{1}\!/_{d}\, \boldsymbol{F} \operatorname{diag}(\hat{\boldsymbol{a}})\, \boldsymbol{F}^{-1} \qquad \text{with} \qquad \boldsymbol{F} = (\mathrm{e}^{-2\pi \mathrm{i} jk/d})_{j,k=0}^{d-1} \quad \text{and} \quad \hat{\boldsymbol{a}} := \boldsymbol{F}\boldsymbol{a}.$$

- We call a kernel $\boldsymbol{a}$ *strictly, symmetrically decreasing* when

$$\hat{\boldsymbol{a}} \in \mathbb{R}^{d}_{++}, \qquad \hat{a}_k = \hat{a}_{-k}, \qquad \text{and} \qquad \hat{a}_k > \hat{a}_j$$

  for $k, j \in \{0, \dots, \lfloor d/2 \rfloor\}$ with $k < j$. [Tang 2017]
- The kernel *collision-free in frequency* if $\hat{a}_j \hat{a}_k$ are unique for $k, j \in \{0, \dots, \lfloor d/2 \rfloor\}$ with $j \geq k$.

Phase Retrieval & System Identification

Theorem (Phase Retrieval & System Identification)

Let $\boldsymbol{a} \in \mathbb{R}^d$ be strictly, symmetrically decreasing and collision-free in frequency, let $\boldsymbol{\phi}_1, \boldsymbol{\phi}_2 \in \mathbb{R}^d$ be pointwise independent in frequency domain, and let $\boldsymbol{x} \in \mathbb{R}^d$ satisfy $\mathfrak{R}[\bar{\hat{x}}_k \hat{\phi}_{i,k}] \neq 0$ for $k = -\lfloor (d-1)/2 \rfloor, \ldots, \lfloor d/2 \rfloor$, $i = 1, 2$. Then $\boldsymbol{a}$ and $\boldsymbol{x}$ can be recovered from the samples

$$\left\{ |\langle x, (\operatorname{circ} \boldsymbol{a})^\ell \boldsymbol{\phi}_1 \rangle|, |\langle x, (\operatorname{circ} \boldsymbol{a})^\ell \boldsymbol{\phi}_2 \rangle| \right\}_{\ell=0}^{L-1} \qquad \text{with} \qquad L := \left(\left\lfloor \tfrac{d}{2} \right\rfloor + 1\right)\left(\left\lfloor \tfrac{d}{2} \right\rfloor + 2\right)$$

up to global sign.

- Let $\xi_{k,j}$ counts the occurrence of $\hat{a}_k \hat{a}_j$. The measurements have the form

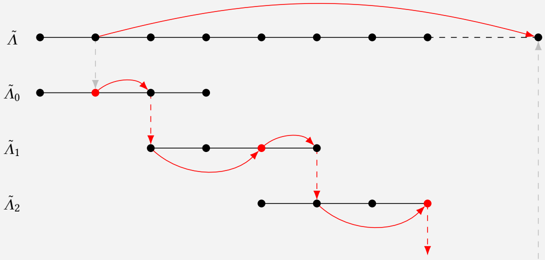
$$|\langle x, (\operatorname{circ} \boldsymbol{a})^\ell \boldsymbol{\phi} \rangle|^2 = \sum_{k=0}^{\lfloor d/2 \rfloor} \sum_{j=k}^{\lfloor d/2 \rfloor} \underbrace{\xi_{k,j} \mathfrak{R}[c_k] \mathfrak{R}[c_j]}_{\eta_k} (\hat{a}_k \hat{a}_j)^\ell = \sum_{k=0}^{L/2-1} \eta_k \beta_k^\ell.$$

What Happens If $\boldsymbol{\phi}_i$ Is Not A-Spectrally Persitent?

- $S^{-1}\boldsymbol{\phi}_i$ might have some zero coordinates.
- The temporal samples become

$$|\langle x, A^\ell \boldsymbol{\phi}_i \rangle|^2 = |\langle \boldsymbol{y}, \boldsymbol{\Lambda}^\ell \boldsymbol{\psi}_i \rangle|^2 = \Bigg| \sum_{k \in \mathcal{I}_i} \lambda_k^\ell \underbrace{\bar{y}_k \psi_{i,k}}_{=: c_{i,k}} \Bigg|^2 = \sum_{j,k \in \mathcal{I}_i} c_{i,j} \bar{c}_{i,k} \, (\lambda_j \bar{\lambda}_k)^\ell,$$

  where $\boldsymbol{y} := S^* x$, $\boldsymbol{\psi}_i := S^{-1}\boldsymbol{\phi}_i$, and $\mathcal{I}_i := \operatorname{supp} \boldsymbol{\psi}_i$.
- Idea behind the reconstruction:



Definition (Parameter Identification & Phase Retrieval)

We say that the pair $(\{\boldsymbol{\phi}_i\}_{i=0}^{J-1}, \boldsymbol{S})$ with $\boldsymbol{\psi}_i := \boldsymbol{S}^{-1}\boldsymbol{\phi}_i$ allows

(i) *index separation* if $\bigcup_{i=0}^{J-1} \operatorname{supp} \boldsymbol{\psi}_j = \{0, \dots, d-1\}$, and if there exist $\mathcal{F}_k$ and $\mathcal{G}_k$ such that

$$\{k\} = \bigcap_{i \in \mathcal{F}_k} \operatorname{supp} \boldsymbol{\psi}_i \Big\backslash \bigcup_{i \in \mathcal{G}_k} \operatorname{supp} \boldsymbol{\psi}_i \quad \text{for every} \quad k \in \{0, \dots, d-1\},$$

(ii) *phase propagation* if the set $\{\boldsymbol{\phi}_i\}_{i=0}^{J-1}$ is ordered such that

$$\#\left[\operatorname{supp} \boldsymbol{\psi}_k \cap \bigcup_{i=0}^{k-1} \operatorname{supp} \boldsymbol{\psi}_i\right] \geq 2 \quad \text{for} \quad k = 0, \dots, d-1,$$

(iii) *winding direction determination* if there are indices i_1, i_2, k_1, k_2 such that

$$\arg(\psi_{i_1,k_1}\bar{\psi}_{i_1,k_2}) \not\equiv \arg(\psi_{i_2,k_1}\bar{\psi}_{i_2,k_2}) \bmod \pi,$$

where $\psi_{i_1,k_1}\bar{\psi}_{i_1,k_2}$ and $\psi_{i_2,k_1}\bar{\psi}_{i_2,k_2}$ are non-zero.

Theorem (Multiple Sampling Vectors)

Let $A = S\Lambda S^{-1}$ be diagonalizable by a known eigenvector basis S and assume that the eigenvalues are absolutely collision-free. Let $(\{\boldsymbol{\phi}_j\}_{j=0}^{J-1}, S)$ allow parameter identification and phase retrieval, and let $\boldsymbol{y} := S^* \boldsymbol{x}$ be elementwise non-zero for unknown $\boldsymbol{x} \in \mathbb{C}^d$. Then the eigenvalues $\lambda_0, \dots, \lambda_{d-1}$ of A and the signal $\boldsymbol{x}$ are determined by the spatiotemporal samples

$$\left\{ |\langle x, A^\ell \boldsymbol{\phi}_i \rangle| \right\}_{\ell, i=0}^{L_i^2 - 1, J-1} \qquad \text{with} \qquad L_i := \#[\operatorname{supp}(S^{-1} \boldsymbol{\phi}_i)]$$

up to global phase.

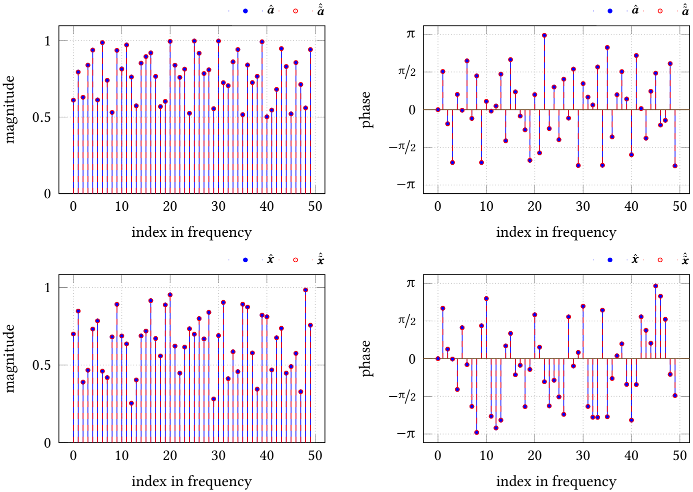
Number of Measurements

- The number of samples mainly correlate with $L_i := \#[\operatorname{supp}(S^{-1} \boldsymbol{\phi}_i)]$.
- With $L := \max\{L_i : i = 0, \dots, J-1\}$, the number of samples is thus bounded by $2L^2 J$.

Corollary (Number of Measurements)

The eigenvalues of $A \in \mathbb{C}^{d \times d}$ and the unknown signal $\boldsymbol{x} \in \mathbb{C}^d$ are identifiable with $\mathcal{O}(d)$ spatiotemporal samples.

Numerical Example



- Generalize the results of [Potts & Tasche 2010] to arbitrary exponential sums.
- Define

$$\pi_{\boldsymbol\beta} := \prod_{k=0}^{K-1}(1+|\beta_k|), \qquad \sigma_{\boldsymbol\beta} := \min\{|\beta_\ell-\beta_k|\}, \qquad \rho_{\boldsymbol\beta} := \max\{1, \|\boldsymbol\beta\|_\infty\}.$$

Theorem (Approximate Bases)

Let $\tilde{\boldsymbol\gamma}$ be the right singular vector to the smallest singular value of $\tilde{\boldsymbol H} := \left(h_{\ell+k}+e_{\ell+k}\right)_{\ell,k=0}^{L-K-1,K}$. Then the corresponding polynomial $\tilde P(z) = \sum_{k=0}^{K} \tilde\gamma_k z^k$ satisfies

$$\sum_{k=0}^{K-1} |\tilde P(\beta_k)|^2 \le KL\,\rho_{\boldsymbol\beta}^{2L-2}\,\frac{\left(\tilde\sigma_K + \|\boldsymbol E\|_2\right)^2}{\sigma_{K-1}^2}.$$

Theorem (Approximate Coefficients)

The least-squares solution $\tilde{\boldsymbol\eta}$ of $V_L(\tilde{\boldsymbol\beta})\,\tilde{\boldsymbol\eta} = \tilde{\boldsymbol h}$ with $\|\boldsymbol h-\tilde{\boldsymbol h}\|_\infty \le \epsilon$, $\|\boldsymbol\beta-\tilde{\boldsymbol\beta}\|_\infty \le \delta$, and $\delta < \sigma_{\boldsymbol\beta}/2$ satisfies

$$\|\boldsymbol\eta-\tilde{\boldsymbol\eta}\|_\infty \le \frac{\pi_{|\boldsymbol\beta|+\delta 1}}{(\sigma_{\boldsymbol\beta}-2\delta)^{K-1}}\left(\sqrt{2}\,KL\,\frac{\pi_{\boldsymbol\beta}\,\rho_{|\boldsymbol\beta|+\delta 1}^{L-1}}{\sigma_{\boldsymbol\beta}^{K-1}}\|\boldsymbol h\|_\infty\,\delta + \epsilon\right).$$

Proposition (Error Spectrum)

Assume $\| \tilde{\boldsymbol{\beta}} - \boldsymbol{\beta} \|_\infty \leq \delta$, and estimate $\boldsymbol{\lambda}$ by using phase propagation. If $\delta < \| \boldsymbol{\lambda} \|_{-\infty}^2$, then we have

$$\|\tilde{\boldsymbol{\lambda}} - \boldsymbol{\lambda}\|_\infty \leq \left(\frac{2\sqrt{2}}{\|\boldsymbol{\lambda}\|_{-\infty}} + \frac{1}{2\sqrt{\|\boldsymbol{\lambda}\|_{-\infty}^2 - \delta}} \right) \delta.$$

Proposition (Error Coefficients)

Assume $\| \tilde{\boldsymbol{\eta}} - \boldsymbol{\eta} \|_\infty \leq \epsilon$, and estimate $\boldsymbol{y}$ by phase propagation. If $\epsilon < \| \boldsymbol{y} \|_{-\infty}^2 \| \boldsymbol{\psi} \|_{-\infty}^2$, then we have

$$\|\tilde{\boldsymbol{y}} - \boldsymbol{y}\|_\infty \leq \left(\frac{2\sqrt{2} \, \|\boldsymbol{y}\|_\infty \, \|\boldsymbol{\psi}\|_\infty}{\|\boldsymbol{y}\|_{-\infty}^2 \, \|\boldsymbol{\psi}\|_{-\infty}^2} + \frac{1}{2 \, \|\boldsymbol{\psi}\|_{-\infty} \sqrt{\|\boldsymbol{y}\|_{-\infty}^2 \, \|\boldsymbol{\psi}\|_{-\infty}^2 - \epsilon}} \right) \epsilon$$

and thus

$$\|\tilde{\boldsymbol{x}} - \boldsymbol{x}\|_\infty \leq \left(\frac{2\sqrt{2} \, \|\boldsymbol{y}\|_\infty \, \|\boldsymbol{\psi}\|_\infty}{\|\boldsymbol{y}\|_{-\infty}^2 \, \|\boldsymbol{\psi}\|_{-\infty}^2} + \frac{1}{2 \, \|\boldsymbol{\psi}\|_{-\infty} \sqrt{\|\boldsymbol{y}\|_{-\infty}^2 \, \|\boldsymbol{\psi}\|_{-\infty}^2 - \epsilon}} \right) \|S^{-1}\|_1 \, \epsilon.$$

Summary/Outlook

- Exclusive and simultaneous phase retrieval and system identification.
- Constructive proofs and algebraic recovery methods.
- Reduction of needed samples to $\mathcal{O}(L^2 d)$.
- Sensitivity analysis of Prony's method and phase propagation.

- Do we really need $2d^2$ samples for Prony's method?
- Can we do similar constructions in infinite dimensions?

Thank you for the attention.

- Robert Beinert, Marzieh Hasannasab. Phase retrieval and system identification in dynamical sampling via Prony's method, 2021.