Tensor-Free Algorithms for Convex Liftings of Bilinear Inverse Problems with Applications to Masked Phase Retrieval

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Masked Fourier phase retrieval

Problem formulation

Recover the image $\boldsymbol{u} \in \mathbb{C}^{N \times N}$ from its *masked Fourier intensities*

$$|\mathcal{F}_M[\boldsymbol{d}_\ell \odot \boldsymbol{u}]| \qquad (\ell = 0, \dots, L-1). \qquad (\mathfrak{F})$$

- More precisely, the unknown signal $\boldsymbol{u}$ is first pointwise multiplied with the masks $\boldsymbol{d}_\ell \in \mathbb{C}^{N \times N}$ and afterwards M-point Fourier transformed by

$$(\mathcal{F}_M[\boldsymbol{v}])[m_2, m_1] := \sum_{n_2, n_1 = 0}^{N-1} \boldsymbol{v}[n_2, n_1] \mathrm{e}^{-2\pi\mathrm{i}(n_2 m_2 + n_1 m_1)/M}.$$

- Taking squares, the problem becomes $|\mathcal{F}_M[\boldsymbol{u}]|^2 = g^\dagger$ and may be interpreted as *quadratic inverse problem*, which is a special case ($\boldsymbol{u} = \boldsymbol{v}$) of the *bilinear inverse problems.*

Bilinear inverse problem

Let $\mathcal{B} \colon \mathcal{H}_1 \times \mathcal{H}_2 \to \mathcal{K}$ be a bilinear operator between finite-dimensional, real Hilbert spaces. Recover $(\boldsymbol{u}, \boldsymbol{v})$ from the given data

$$\mathcal{B}(\boldsymbol{u}, \boldsymbol{v}) = \boldsymbol{g}^\dagger. \qquad (\mathfrak{B})$$

First-order proximal algorithms for linear liftings

Tensorial liftings

Using the universal property, we always find a unique linear *tensorial lifting* $\breve{\mathcal{B}} \colon \mathcal{H}_1 \otimes \mathcal{H}_2 \to \mathcal{K}$ with $\breve{\mathcal{B}}(u \otimes v) = \mathcal{B}(u, v)$.

- Therefore, the bilinear inverse problem $(\mathfrak{B})$ is equivalent to

$$\breve{\mathcal{B}}(\boldsymbol{w}) = \boldsymbol{g}^\dagger \qquad \text{with} \qquad \operatorname{rank}(\boldsymbol{w}) = 1. \qquad (\breve{\mathfrak{B}})$$

- To find the solution in the noise-free case, we minimize the nuclear norm $\|\cdot\|_{\mathcal{H}_1 \otimes_\pi \mathcal{H}_2}$ over the solution set of the lifted problem.

Convex relaxation

Solve the convex relaxed version of $(\breve{\mathfrak{B}})$:

$$\text{minimize} \quad \|\boldsymbol{w}\|_{\mathcal{H}_1 \otimes_\pi \mathcal{H}_2} \quad \text{with} \quad \breve{\mathcal{B}}(\boldsymbol{w}) = \boldsymbol{g}^\dagger. \qquad (\mathfrak{B}_0)$$

- Minimizers for $(\mathfrak{B}_0)$ can be determined by first-order proximal algorithms like the primal-dual iteration [CP11].

Primal-dual iteration

For suitable $\sigma, \tau > 0$, and $\theta \in [0, 1]$, the iteration

$$\boldsymbol{y}^{(n+1)} := \boldsymbol{y}^{(n)} + \sigma \left(\breve{\mathcal{B}}\left((1+\theta)\, \boldsymbol{w}^{(n)} - \theta \boldsymbol{w}^{(n-1)} \right) - \boldsymbol{g}^\dagger \right),$$

$$\boldsymbol{w}^{(n+1)} := \mathcal{S}_\tau\left(\boldsymbol{w}^{(n)} - \tau\, \breve{\mathcal{B}}^*(\boldsymbol{y}^{(n+1)}) \right).$$

yields a solution of the convex relaxation $(\mathfrak{B}_0)$.

- The *(soft) singular value thresholding* is given by

$$\mathcal{S}_\tau(\boldsymbol{w}) := \sum_{r=0}^{R-1} S_\tau(\sigma_r)\, (\boldsymbol{u}_r \otimes \boldsymbol{v}_r) \quad \text{with} \quad S_\tau(t) := \begin{cases} t - \tau & \text{if } t \geq \tau, \\ 0 & \text{if } t < \tau, \end{cases}$$

where $\boldsymbol{w} = \sum_{r=0}^{R-1} \sigma_r\, (\boldsymbol{u}_r \otimes \boldsymbol{v}_r)$ is the SVD with respect to $\mathcal{H}_1$ and $\mathcal{H}_2$.

- Adaptations to quadratic problems like phase retrieval and to noisy measurements are straightforward.

Tensor-free computation and reweighting

- The main drawbacks of the proposed algorithm are the tensorial operations $\mathcal{S}_\tau$, $\breve{\mathcal{B}}$, and $\breve{\mathcal{B}}^*$, especially, due to the dimension of $\mathcal{H}_1 \otimes \mathcal{H}_2$.
- To overcome this issue, we exploit that the nuclear norm heuristic usually enforces low-rank iterations.

Tensor-free proximal mappings

- Storing the singular values/vectors instead of the full tensor, we can efficiently evaluate the operator $\breve{\mathcal{B}}$ using the universal property.
- For $\mathcal{S}_\tau(\boldsymbol{w})$, we apply a restarted augmented Lanczos process [BR05] to compute all singular values/vectors greater than τ. This process only require the left- and right-hand action of $\boldsymbol{w}$.
- The actions of $\breve{\mathcal{B}}^*(\boldsymbol{y}^{(n+1)})$ can be computed using the adjoints of the linear mappings $\boldsymbol{u} \mapsto \mathcal{B}(\boldsymbol{u}, \boldsymbol{v}')$, $\boldsymbol{v} \mapsto \mathcal{B}(\boldsymbol{u}', \boldsymbol{v})$ for fixed $\boldsymbol{u}' \in \mathcal{H}_1$, $\boldsymbol{v}' \in \mathcal{H}_2$.
- For further rank reduction, we propose a novel reweighting heuristic for $\mathcal{H}_1$ and, analogously, for $\mathcal{H}_2$.

Hilbert space reweighting

Every few iterations, replace the inner product of $\mathcal{H}_1$ by

$$\langle \boldsymbol{u}, \boldsymbol{v} \rangle_{\mathcal{H}_1'} := \langle \boldsymbol{u}, \boldsymbol{v} \rangle_{\mathcal{H}_1} - \sum_{n=0}^{S-1} \lambda_n \langle \boldsymbol{u}, \boldsymbol{e}_n \rangle_{\mathcal{H}_1} \langle \boldsymbol{e}_n, \boldsymbol{v} \rangle_{\mathcal{H}_1}$$

with $\lambda_n \in (0, 1)$ and orthonormal vectors $\boldsymbol{e}_n$.

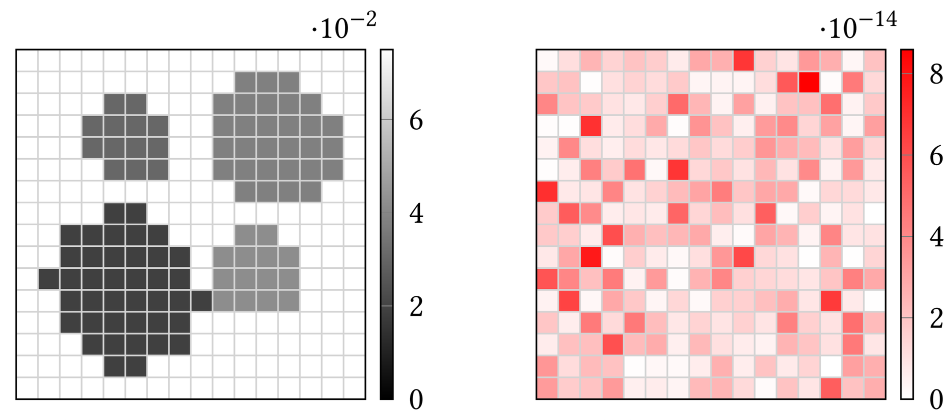
- Choosing $\boldsymbol{e}_n$ as the singular vectors of $\boldsymbol{w}^{(n)}$, we can promote these directions in the subsequent iterations.

Application to masked phase retrieval

- We apply the developed algorithm to the phase retrieval problem $(\mathfrak{F})$ with randomly generated Rademacher masks

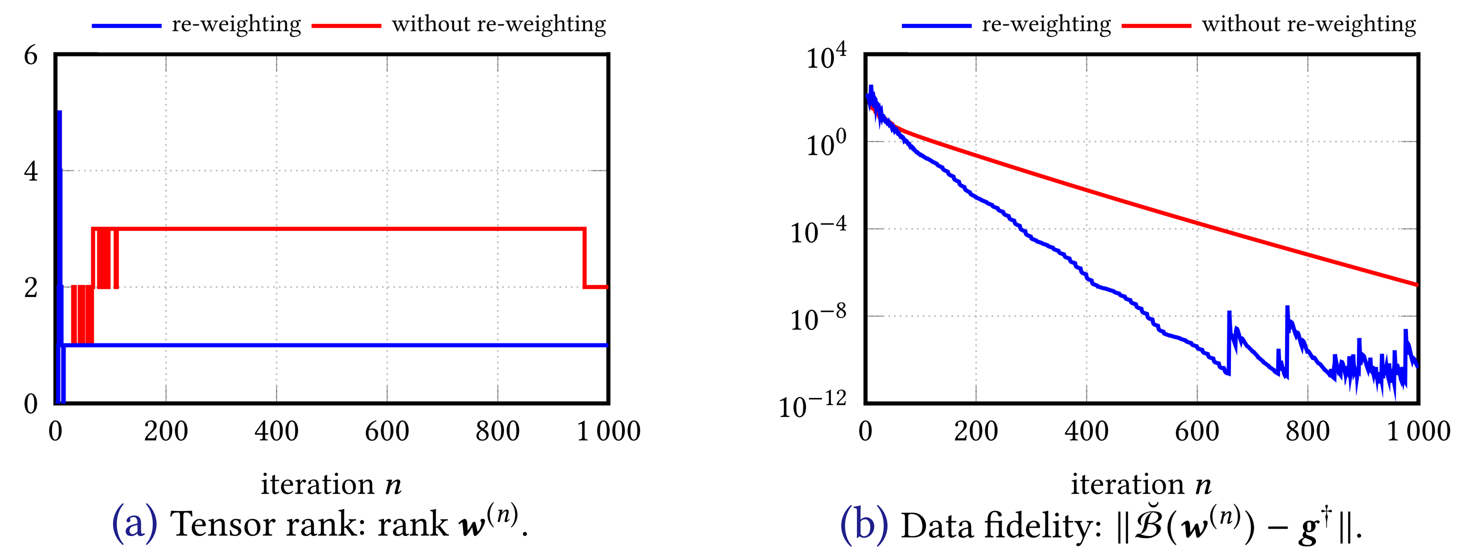
$$\boldsymbol{d}_\ell[n_2, n_1] \sim \begin{cases} \sqrt{2} & \text{with probability } 1/4, \\ 0 & \text{with probability } 1/2, \\ -\sqrt{2} & \text{with probability } 1/4, \end{cases}$$

see [CLS15, GKK17].



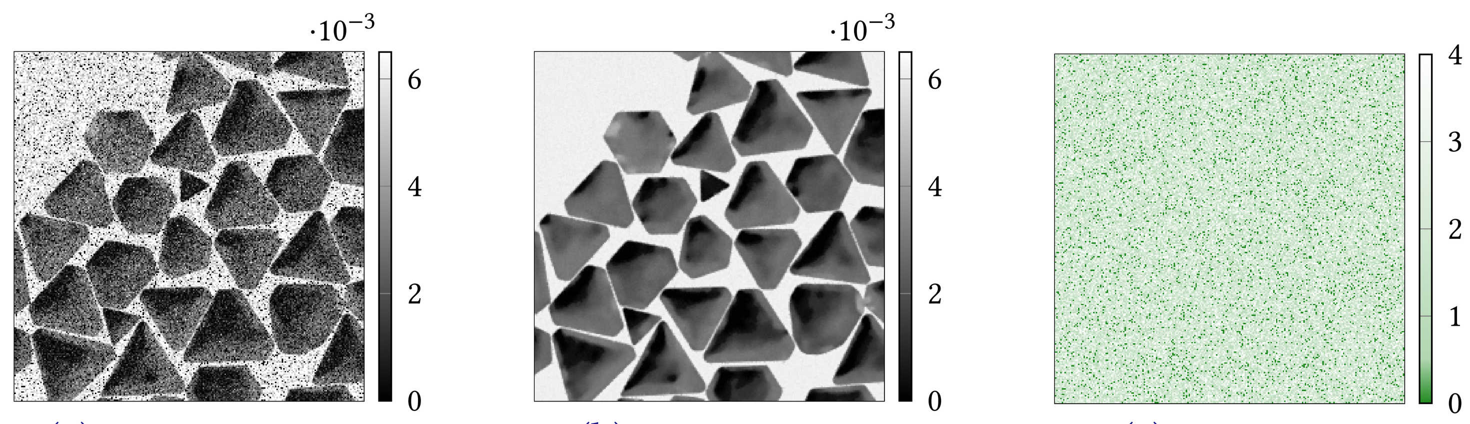
(a) Recovered signal. (b) Absolute error.

Figure: Primal-dual iteration for a synthetic test image with augmented Lanczos process and reweighting.



(a) Tensor rank: rank $\boldsymbol{w}^{(n)}$. (b) Data fidelity: $\|\breve{\mathcal{B}}(\boldsymbol{w}^{(n)}) - \boldsymbol{g}^\dagger\|$.

Figure: Evolution of the rank and data fidelity with and without reweighting in the previous example.



(a) Euclidian setting. (b) Sobolev setting. (c) Blocked pixel.

Figure: Results for a 256×256 test image. (Fourier data created on the basis of TEM micrographs of gold nanoparticles [LSP+16].) The Rademacher masks block a sixteenth of the image. Here, we endow $\mathcal{H}_1$ and $\mathcal{H}_2$ with the Euclidian and with a Sobolev inner product.

Conclusion

- Using the augmented Lanczos process and reweighting technique, we obtain an efficient lifting algorithm that avoids direct computations on the tensor product.
- Choosing $\mathcal{H}_1$ and $\mathcal{H}_2$ suitably, we can directly incorporate a priori assumptions like smoothness properties into the nuclear norm.
- The proposed recovery technique may also be applied to further bilinear or quadratic recovery problems like deautoconvolution, blind deconvolution, and parallel magnetic resonance imaging.

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Referenzen

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