Ambiguities in one-dimensional phase retrieval

Robert Beinert, Gerlind Plonka

Institut für Numerische und Angewandte Mathematik

Georg-August-Universität Göttingen

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Formulation of the discrete-time problem

In many scientific areas, such as astronomy, electron microscopy, and crystallography, one is faced with the problem to recover an unknown signal from the magnitudes of its Fourier transform.

Problem (Discrete-time phase retrieval). Recover the unknown *complex-valued* signal $x := (x[n])_{n \in \mathbb{Z}}$ with *finite support* from its Fourier intensities

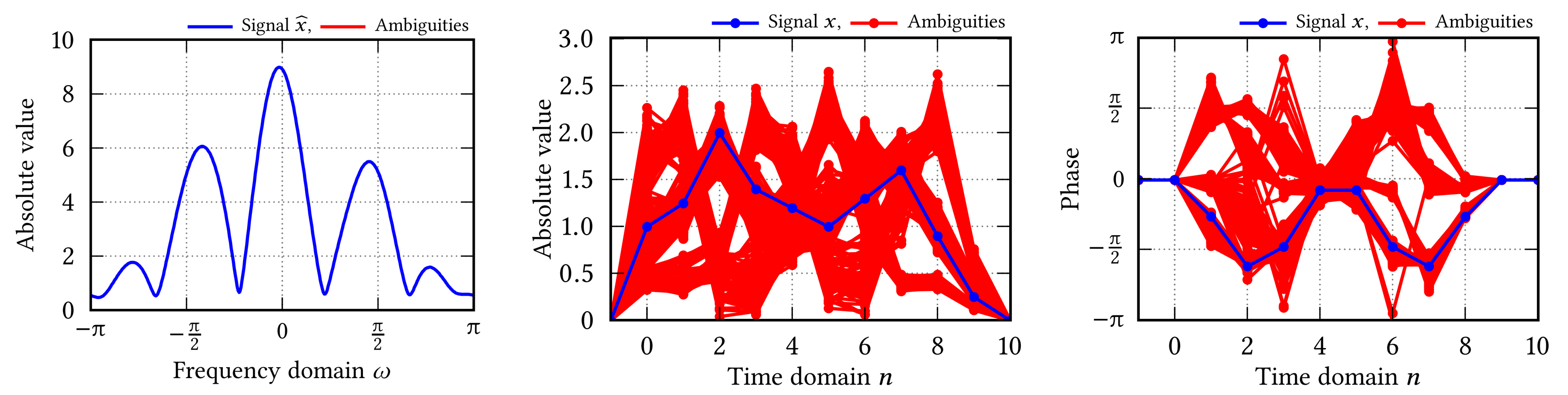
$$|\hat{x}(\omega)| := \left| \sum_{n \in \mathbb{Z}} x[n] \, \mathrm{e}^{-\mathrm{i}\omega n} \right| \qquad (\omega \in \mathbb{R}).$$

Trivial and non-trivial ambiguities

Unfortunately, this phase retrieval problem is complicated because of the well-known ambiguousness. For example,

- the *rotated* signal $(y[n]) := (\mathrm{e}^{\mathrm{i}\alpha} x[n])$
- the *shifted* signal $(y[n]) := (x[n - n_0])$
- the *reflected, conjugated* signal $(y[n]) := (\overline{x[-n]})$

have the same Fourier intensity $|\hat{x}|$ as the original signal x. Besides these *trivial ambiguities*, the discrete-time phase retrieval problem usually possesses additional *non-trivial ambiguities*.



(a) Fourier intensity: $|\hat{x}(\omega)|$ **(b)** Absolute value: $|x[n]|$ **(c)** Phase: $\arg(x[n])$

Figure: Discrete-time phase retrieval problem with 2^8 non-trivial ambiguities

The phase retrieval problem in the frequency domain

Definition (Autocorrelation). The *autocorrelation signal a* of the signal x is defined by

$$a[n] := \sum_{k \in \mathbb{Z}} \overline{x[k]} \, x[k+n] \qquad (n \in \mathbb{Z}).$$

The *autocorrelation function A* is the Fourier transform of the autocorrelation signal a.

Obviously, the autocorrelation signal is *conjugate symmetric*. Since x is finitely supported, the autocorrelation function is always a real trigonometric polynomial. Further, the relation to the phase retrieval problem is given by

$$|\hat{x}(\omega)|^2 = \left(\sum_{n \in \mathbb{Z}} x[n] \, \mathrm{e}^{-\mathrm{i}\omega n} \right) \left(\sum_{k \in \mathbb{Z}} \overline{x[k]} \, \mathrm{e}^{\mathrm{i}\omega k} \right) = \sum_{n \in \mathbb{Z}} \sum_{k \in \mathbb{Z}} x[k+n] \, \overline{x[k]} \, \mathrm{e}^{-\mathrm{i}\omega n} = A(\omega).$$

Equivalent problem. Let A be a non-negative trigonometric polynomial. Find a complex trigonometric polynomial B such that

$$|B(\omega)|^2 = A(\omega) \qquad (\omega \in \mathbb{R}).$$

Characterization of the solution set

Using a construction of Fejér and Riesz [4], which is based on the *associated polynomial*

$$P_A(z) := \sum_{n=0}^{2N-2} a[n - N + 1] \, z^n,$$

whose zeros appear in reflected pairs $(\gamma_j, \overline{\gamma_j}^{-1})$, we can characterize all possible solutions of the equivalent problem in the frequency domain and hence of the discrete-time phase retrieval problem.

Theorem (Beinert, Plonka (2015)). *Let A be a non-negative trigonometric polynomial. Then the problem $|B|^2 = A$ possesses at least one solution and each solution is of the form*

$$B(\omega) = \mathrm{e}^{\mathrm{i}\alpha + \mathrm{i}\omega n_0} \sqrt{|a[N-1]| \prod_{j=1}^{N-1} |\beta_j|^{-1}} \cdot \prod_{j=1}^{N-1} \left(\mathrm{e}^{-\mathrm{i}\omega} - \beta_j \right),$$

where β_j can be chosen from the zero pair $(\gamma_j, \overline{\gamma_j}^{-1})$ of the polynomial P_A.

Retransforming this characterization, we can moreover show that each solution can be represented by an appropriate convolution and suitable rotations, shifts, and reflections/conjugations of the occurring factors.

Theorem (Beinert, Plonka (2015)). *Let the signal x with finite support be given by the factorization*

$$(x_1 * x_2)[n] = \sum_{k \in \mathbb{Z}} x_1[k] \, x_2[n-k].$$

Then the signal

$$y := \mathrm{e}^{\mathrm{i}\alpha} \left(\overline{x_1[-\cdot]} \right) * (x_2[\cdot - n_0])$$

has the same Fourier intensity $|\hat{x}|$. Moreover, each signal with the Fourier intensity $|\hat{x}|$ can be represented in this manner.

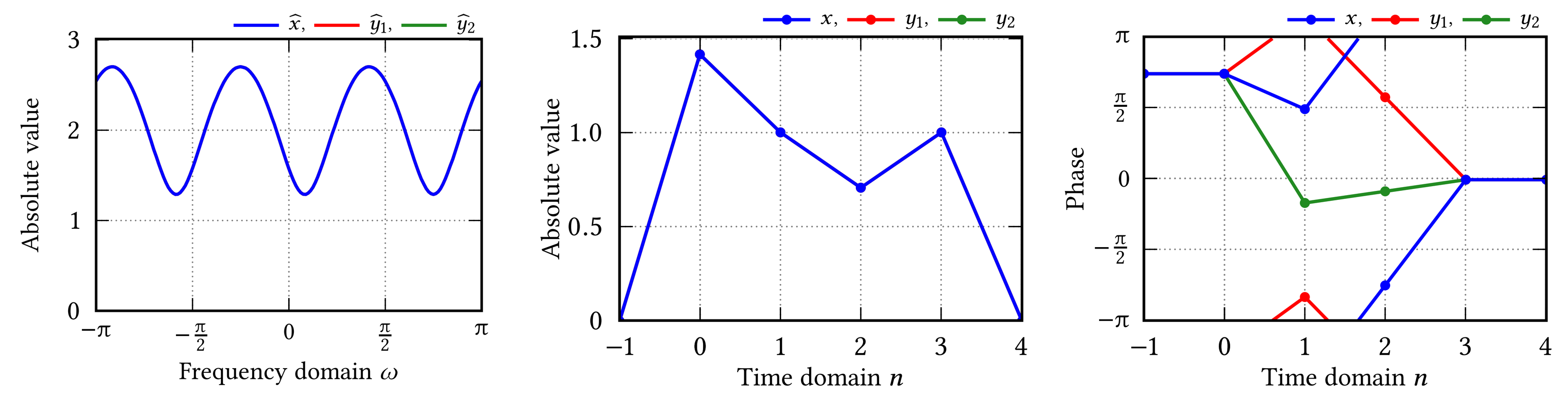
Enforcing uniqueness by additional data in the time domain

In order to find the original signal within the solution set, we require further information about the unknown signal. Assuming that x has the support $\{0, \dots, N-1\}$, we firstly consider the case where one additional modulus $|x[\ell]|$ for an ℓ within the support is available.

Theorem (Beinert, Plonka (2015)). *Almost every signal x can be uniquely recovered up to rotations from its Fourier intensity $|\hat{x}|$ and the modulus $|x[\ell]|$ with $\ell \neq {}^{(N-1)}/_2$.*

For the special case $\ell = {}^{(N-1)}/_2$, the recovery is only unique up to reflection/conjugation and rotations.

Unfortunately, the additional knowledge of more than one or even all moduli in the time domain cannot ensure the unique recovery of every possible signal as shown in the following specific example.



(a) Fourier intensity: $|\hat{x}(\omega)|$ **(b)** Absolute value: $|x[n]|$ **(c)** Phase: $\arg(x[n])$

Figure: Discrete-time phase retrieval problem to recover the signal x from its Fourier intensity $|\hat{x}|$ and the complete modulus $|x|$ with two further non-trivial ambiguities y_1 and y_2.

If we now replace the known modulus in the time domain with the phases $\arg x[\ell_1]$ and $\arg x[\ell_2]$ for appropriate numbers ℓ_1 and ℓ_2 within the support, we obtain a similar result.

Theorem (Beinert (2015)). *Almost every signal x can be uniquely recovered from its Fourier intensity $|\hat{x}|$ and the phases $\arg x[\ell_1]$ and $\arg x[\ell_2]$ with $\ell_1 + \ell_2 \neq N - 1$.*

For the special case where $\ell_1 + \ell_2 = N - 1$ and where ℓ_1 and ℓ_2 do not coincide with both end points, the recovery is only unique up to reflection/conjugation.

References

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