Technische Universität Berlin – Institut für Mathematik SoSe 2016
Dr. Kersten Schmidt

Asymptotic Analysis

Series 1

1. Show:

a) $\tan(x) = x + o(x^2)$ for $x \to 0$ without using the Taylor series for $\tan(x)$
b) $e^{o(x)} = 1 + o(x)$ for $x \to 0$ without any condition on the smoothess of $o(x)$ in x
c) $o(f(x)g(x)) = f(x)o(g(x))$ for $x \to x_0$

2. Prove or disprove:

$$(f = O(g), g = O(f) \text{ for } x \to x_0) \Rightarrow (\exists K \neq 0 : f \sim Kg \text{ for } x \to x_0)$$

3. Order the following functions in an asymptotic sequence for $\varepsilon \to 0$:

$$\varepsilon^2, \sqrt{|\varepsilon|}, \log\left|\log\frac{1}{|\varepsilon|}\right|, 1, \sqrt{|\varepsilon|}\log\frac{1}{|\varepsilon|}, \varepsilon\log\frac{1}{|\varepsilon|}, \log\frac{1}{|\varepsilon|}, |\varepsilon|^{3/2}, \varepsilon^2\log\frac{1}{|\varepsilon|}$$

4. a) Given $v_j \in \mathbb{C}, v_j \neq 0$ such that we have the asymptotic expansion

$$v^{\varepsilon,N} \sim \sum_{j=0}^{N} \varphi_j(\varepsilon) v_j$$

of v^ε for $\varepsilon \to 0$. Show that, for any $N \in \mathbb{N}$ and any $C \leqslant 1$, that there exists a constant $\varepsilon_{N,C}$ such that, for any $\varepsilon < \varepsilon_{N,C}$,

$$\left\|v^{\varepsilon,N+1} - v^\varepsilon\right\| \leqslant C \left\|v^{\varepsilon,N} - v^\varepsilon\right\| \tag{1}$$

b) Given the expansion

$$\int_0^\infty \frac{e^{-\frac{t}{\varepsilon}}}{1+t} dt = \sum_{n=0}^{\infty} (-1)^n n! \varepsilon^{n+1}$$

Specify the constant $\varepsilon_{N,C}$ such that (1) holds.

See next page!

5. a) Consider the solution of the two boundary value problem

$$\begin{cases} \varepsilon u_\varepsilon''(x) + u_\varepsilon'(x) = -sin(x), \quad 0 < x < \pi, \\ u_\varepsilon(0) = u_\varepsilon(\pi) = 0. \end{cases} \tag{2}$$

For a given $x \in (0, \pi)$, make a Taylor expansion of the solution with respect to ε. Show then that the j-th term of the expansion $u_{j,\varepsilon}$ does not depend on ε (hint: show that for $\exp(-x\varepsilon^{-1})$ is $o(\varepsilon^n)$, for any $n \in \mathbb{N}$.

Show that the approximation

$$u^{\varepsilon,N} = \sum_{j=0}^{N} \varepsilon^j u_{j,\varepsilon}$$

is not convergent in $\mathrm{H}^1(0, \pi)$ (hint: prove that the H^1 semi-norm of u_ε is not bounded, but the H^1 of $u^{\varepsilon,N}$ is).

b) Show that there exists a family $\tilde{u}_{j,\varepsilon}$ such that the approximation

$$\tilde{u}^{\varepsilon,N} = \sum_{j=0}^{N} \varepsilon^j \tilde{u}_{j,\varepsilon}$$

is convergent and is an asymptotic expansion in $\mathrm{H}^1(0, \pi)$.

c) Show that the previous approximation is NOT convergent when ε is large enough.

To be handed in by: May 12th, 2016 (2.15 pm, before lecture starts)

This exercise series will be discussed in the tutorial class on May 18th, 2016, 2:15 p.m.

There will be no tutorial class on May 4th. Instead there will be a voluntary consultation in MA 376 starting at 2:15 p.m.

Website: `http://www.tu-berlin.de/?asymptotic-analysis`

Coordinator:

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