Technische Universität Berlin – Institut für Mathematik SoSe 2016
Dr. Kersten Schmidt

ASYMPTOTIC ANALYSIS

Series 4

1. Let $a, b \in \mathbb{R}$, $f \in \mathcal{C}^1([a,b])$, $h \in \mathcal{C}^N([a,b])$, $f(b) \neq 0$, $h'(b) = \cdots = h^{(N-1)}(b) = 0$, $h^{(N)} \neq 0$, $N \geqslant 2$, $h'(t) \neq 0$ for any $a \leqslant t < b$. Prove that

$$\int_a^b e^{\mathrm{i}xh(t)} f(t)dt = \frac{1}{N}\Gamma\left(\frac{1}{N}\right) f(b) e^{\mathrm{i}xh(b)} \left(\frac{(-1)^N N! \mathrm{i}}{xh^{(N)}(b)}\right)^{1/N} + O(1/x), \quad x \to \infty$$

Hint: we recall that, for $0 \leqslant \alpha < 1$ and for $x > 0$, that we have

$$\int_0^\infty e^{\mathrm{i}xt} t^{\alpha-1} dt = \Gamma(\alpha) \left(\frac{\mathrm{i}}{x}\right)^\alpha$$

2. Show that

$$\int_0^{\pi/2} e^{\mathrm{i}x\sin(t)} dt \sim \sqrt{\frac{\pi}{2x}} e^{\mathrm{i}\left(x - \frac{\pi}{4}\right)}, \quad x \to \infty$$

3. Given the solution

$$u_\varepsilon(x) = \frac{1}{1+\varepsilon^2}\left(1 + \cos(x) + \varepsilon \sin(x) - 2\frac{e^{-x/\varepsilon} - e^{-\pi/\varepsilon}}{1 - e^{-\pi/\varepsilon}}\right)$$

of the singular perturbed differential equation

$$\begin{cases} \varepsilon u_\varepsilon''(x) + u_\varepsilon'(x) = -\sin(x), \quad 0 < x < \pi, \\ u_\varepsilon(0) = u_\varepsilon(\pi) = 0. \end{cases} \tag{1}$$

show that the convergence

$$u_\varepsilon(x) \to u_0(x) := 1 + \cos(x)$$

is *not* uniform on $(0, \pi)$.

See next page!

To be handed in by: June 23th, 2016 (2.15 pm, before lecture starts)

This exercise series will be discussed in the tutorial class on June 29th, 2016, 2.15 p.m. in MA 376.

Website: `http://www.tu-berlin.de/?asymptotic-analysis1`

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