Technische Universität Berlin – Institut für Mathematik SoSe 2012
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FEM for Electromagnetics and Wave Propagation

Series 3

1. $H^1(\Omega)$-ellipticity Poisson problem with Robin boundary conditions

Let u be the solution of

$$\begin{aligned} -\Delta u(\mathbf{x}) &= f(\mathbf{x}), && \text{in } \Omega, \\ \partial_n u &= -\tfrac{1}{R} u, && \text{on } \partial\Omega, \end{aligned}$$

where Ω is a bounded domain in $\mathbb{R}^3$ of radius $R > 0$.

a) Write a variational formulation for the above PDE with appropiate trial and test space.

b) Prove the well-posedness of the variational formulation.
Hint: Use Lemma 2.35 of Section 2.9.3.

2. $H^1(\Omega)$-ellipticity for H-formulation for the TE mode in 2D

For an invariant geometry the H-formulation for the TE mode is given by

$$\begin{aligned} -\operatorname{div}(\sigma(\mathbf{x}) - i\omega\varepsilon(\mathbf{x}))^{-1} \,\mathbf{grad}\, u(\mathbf{x}) - \mathrm{i}\omega\mu(\mathbf{x}) u(\mathbf{x}) &= 0, && \text{in } \Omega, \\ u(\mathbf{x}) &= g(\mathbf{x}), && \text{on } \partial\Omega, \end{aligned}$$

where Ω is a bounded Lipschitz domain, u stands for the e_3 component of the magnetic field which is given on the boundary (*e. g.*, by Biot-Savart's law).

a) Write a variational formulation for the above PDE with appropiate subspaces of $H^1(\Omega)$ for trial and test functions. Use test functions $\overline{v}$ to obtain a formulation

$$\mathsf{a}(u, v) = \ell(v)$$

with a sesquilinear form a and an antilinear form ℓ.

b) Show $H^1_0(\Omega)$-ellipticity of the sesquilinear form a if $\mu \equiv 0$ (not physical), $\varepsilon(\mathbf{x}) \geq \varepsilon_0 > 0$ and $\sigma(\mathbf{x}) \geq 0$.

See next page!

c) We would like to discuss the $H^1(\Omega)$-ellipticity of the sesquilinear form a as it applies also to subspaces independant of the boundary conditions (*i. e.*, ignoring Dirichlet boundary conditions at this moment).

Show $H^1(\Omega)$-ellipticity of the sesquilinear form a

(i) if $\varepsilon(\mathbf{x}) \equiv 0$, and if there exists positive constants σ_0 and μ_0 such that $\sigma(\mathbf{x}) \geq \sigma_0 > 0$ and $\mu(\mathbf{x}) \geq \mu_0 > 0$ almost everywhere in Ω,

(ii) if (i) holds, except that there exists constant ε_0 such that $\varepsilon(\mathbf{x}) \geq \varepsilon_0 > 0$ almost everywhere in Ω,

d) Discuss the $H^1(\Omega)$-ellipticity of the sesquilinear form a if **c)**(ii) holds, except that $\sigma \equiv 0$.

e) Is it possible to prove well-posedness of the variational formulation for $\sigma \equiv 0$ if the Dirichlet boundary conditions are used?
Hint: Discuss the uniqueness using the eigenfunctions of the homogeneous Dirichlet eigenvalue problem $-\Delta u = \lambda u$ in Ω and $u = 0$ on $\partial\Omega$.

3. Element stiffness matrix and element load vector for linear finite elements

We are interested in the solution of

$$\begin{aligned} -\operatorname{div}\mathbf{grad}\, u &= f \quad \text{in } \Omega, \\ u &= 0 \quad \text{on } \partial\Omega. \end{aligned} \tag{1}$$

a) Write a function `laplaceP1.m` that returns the local stiffness matrix for a single triangle. The input parameters are the coordinates of the three nodes. Use the analytic formulas derived in the lecture.

b) The function f can be defined as handle like
`f = @(x,y)(x^2 - y^2)` or as a m-file

```
function y = f(x,y)
if x < 0
  return 0;
end
return 1;
```

Formulas for numerical quadratures on triangles are defined on the triangle K_{NQ} with the nodes $(-1,-1)$, $(1,-1)$, $(-1,1)$, see for instance Gauss quadrature rules on page 141 in chapter 4 of the book of P. Solin. They read

$$\int_{K_{NQ}} f(\boldsymbol{\xi})\mathrm{d}\boldsymbol{\xi} \approx \sum_{j=1}^{n} w_j f(\boldsymbol{\xi}_j).$$

See next page!

On the other hand, the element shape functions are defined on the reference triangle $\widehat{K}$ with nodes $(0,0)$, $(1,0)$, $(1,1)$.

Write the linear form

$$\ell_K(v) = \int_K f(\mathbf{x})v(\mathbf{x})\mathrm{d}\mathbf{x} \approx |F_k| \sum_{j=1}^{n} w_j f(\Phi_K \hat{\mathbf{x}}_j)\widehat{N}_j(\hat{\mathbf{x}}_j). \qquad (2)$$

as integral over K_{NQ} (in dependence of the coordinates $\mathbf{p}_0$, $\mathbf{p}_1$, and $\mathbf{p}_2$ of the nodes of K) and using element shape functions $\widehat{N}_j$, which simply transform to K_{NQ}.

Write a function `lastP1.m` that returns an approximation to the element load vector, that is the vector related to the linear form (2) for the three linear shape functions, using Gauss quadrature. The number of quadrature points should be given as input parameter.

```
# Returns the element load vector for
#   int_K f v dx
# for linear FEM on triangles
#
function phi = lastP1(f, p, n)
# input:
# f    function handle to source term
# p    3x2 matrix of the coordinates of the triangle
#      nodes
# n    number of quadrature points
#
# output:
# phi  element load vector (3x1 matrix)

if argin < 3
  n = 1;     # in case of only two input arguments
             # take one quadrature point
end
```

Check your code (at least) with $f = 1$. Note that $\sum_{j=1}^{3} \varphi_j$ for $f = 1$ is equal to the area of the triangle.

To be handed in by: May 22nd, 2012 (1pm–2pm)
Website: `http://www.tu-berlin.de/?maxwell-numerics-lecture`
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