Technische Universität Berlin – Institut für Mathematik SoSe 2012
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FEM for Electromagnetics and Wave Propagation

Series 4

1. A low frequency approximation to linear Maxwell's equations describing the alternating electric and magnetic fields $\mathbf{E}$ and $\mathbf{H}$ in a domain $\Omega \subset \mathbb{R}^3$ is the time-harmonic eddy current model (Wirbelstrommodell)

$$\mathbf{curl}\,\mathbf{E} = \mathrm{i}\omega\mu\mathbf{H}, \tag{1a}$$

$$\mathbf{curl}\,\mathbf{H} = \sigma\mathbf{E}, \tag{1b}$$

where $\omega > 0$ is the (angular) frequency, $\mu \geq \mu_0 > 0$ the permittivity and $\sigma \geq \sigma_0 > 0$ the conductivity. On the boundary $\partial\Omega$ of the domain Ω the tangential component of the electric field is prescribed as

$$\mathbf{E} \times \mathbf{n} = \mathbf{E}_{0,T}. \tag{1c}$$

a) Derive a variational formulation of (1) by taking the weak form of (1a) with test function $\mathbf{H}'$, taking the strong form of (1b) multiplied by a test function $\mathbf{E}' = \sigma^{-1}\,\mathbf{curl}\,\mathbf{H}'$.

b) Is the boundary condition (1c) essential or natural?

c) What is the "ideal" Sobolev space V for the variational formulation?

d) Let $\sigma \equiv \mu \equiv \omega = 1$. Show for the bilinear form b of the variational formulation that there exists a constant $\theta \in [0, 2\pi)$ such that

$$\exists \gamma > 0: \quad \mathrm{Re}\left(e^{\mathrm{i}\theta}\,\mathsf{b}(\mathbf{H}, \overline{\mathbf{H}})\right) \geq \gamma \|\mathbf{H}\|_V^2 \qquad \forall \mathbf{H} \in V,$$

where $\overline{\mathbf{H}}$ is the complex conjugate of $\mathbf{H}$.

2. Geometric optics: Reflection and refraction of plane waves

Consider the (generalized) Helmholtz equation

$$-\operatorname{div} a(\mathbf{x})\,\mathbf{grad}\,u(\mathbf{x}) - b(\mathbf{x})\frac{\omega^2}{c^2}u(\mathbf{x}) = f(\mathbf{x}), \tag{2}$$

where $f = 0$ and a, b correspond to the dielectricity ε depending wether we consider the TE or the TM modes (see Chapter 1). The dielectricities of the two media for $x_2 > 0$ and $x_2 < 0$ are ε_1 and ε_2 respectively.

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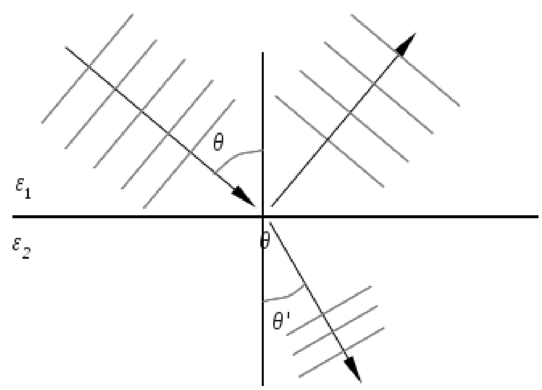
a) Write (2) for the TE and the TM modes.

b) Write the PDE in a) as two PDEs for $x_2 > 0$ and $x_2 < 0$ which are coupled by continuity conditions. Specify those continuity conditions.

c) Now, we have an incoming wave $u^{\text{inc}} = \exp(\mathrm{i}\mathbf{k}_0 \cdot \mathbf{x})$ with $|\mathbf{k}_0| = \frac{\omega}{c}$ and different direction (see the figure below). What are the reflected wave u^{refl} and the refracted wave u^{refr} such that

$$u = \begin{cases} u^{\text{inc}} + u^{\text{refl}} & \text{for } x_2 > 0 \\ u^{\text{refr}} & \text{for } x_2 < 0 \end{cases}$$

fulfills the PDEs in b) with the continuity conditions, both the TM and TE mode. What are the directions and the wavenumbers or the reflected and refracted waves for the two modes?



3. Assembling of stiffness matrix and load vector

We are interested in the solution of

$$\begin{aligned} -\operatorname{div} \mathbf{grad}\, u &= f \quad \text{in } \Omega, \\ u &= 0 \quad \text{on } \partial\Omega. \end{aligned} \tag{3}$$

a) The number of non-zero entries in the $N \times N$ stiffness matrix is much smaller than N^2. Matlab provides a sparse matrix format, see `help sparse`.

Write a function returning the stiffness matrix in sparse format while ignoring the essential boundary condition.

```
# Returns the stiffness matrix for
#   int_Omega grad u . grad v dx
# for linear FEM on triangles
#
function A = stiffnessP1(p, t)
# input:
```

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```
#   p         Nx2 matrix with coordinates of the nodes
#   t         Mx3 matrix with indices of nodes of the
#             triangles
#
# output:
#   A         NxN sparse matrix
```

Use the function `laplaceP1.m` and the relation from local numbering of the shape functions to the global numbering of the basis function given the matrix `t`.

b) Write a function returning the load vector while ignoring the essential boundary condition.

```
# Returns the load vector for
#   int_Omega f v dx
# for linear FEM on triangles
#
function b = rieszTriangleP1(p, t, f)
# input:
#   p         Nx2 matrix with coordinates of the nodes
#   t         Mx3 matrix with indices of nodes of the
#             triangles
#   f
#
# output:
#   b         Nx1 matrix
```

c) To which boundary condition do the stiffness matrix and the load vector — built up to now — correspond?

d) Write a function `boundaryEdges.m` in Matlab that returns a `Bx2` array of boundary edges (indices of the two nodes in increasing order) while taking the indices of the triangles as input.
Hint: Boundary edges appear only in one triangle. So create a matrix with three columns where the first two columns carry the node indices of the edges and the last column gives the number of triangles the edge appears in. Finally, delete the rows with double appearance and delete the last column.

e) Using the boundary edges determine the reduced index set `IR` belonging to interior nodes.
Hint: Use the Matlab/Octave functions `setdiff` *and* `unique`.

f) Write the main code for solving (3) including

- Choosing a geometry (e.g. square).

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- Mesh generation for the chosen geometry.
- Computation of the stiffness matrix and load vector ignoring the essential boundary condition.
- Solving the reduced system with `u(IR) = A(IR,IR)\b(IR)`, where `u` has to be initialised with zeros to fix the boundary values.
- Graphical representation with `trimesh` or `trisurf`.

Check the code for the square $\Omega = (0,1)^2$ and a function f such that $u = \sin(\pi x_1)\sin(\pi x_2)$ solves (3).

To be handed in by: June 5th, 2012
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