INTRODUCTION TO THE FINITE ELEMENT METHOD

Project 5 — FE Solver for Parabolic PDEs

1. Temporal discretization of parabolic partial differential equations

Let us consider the transient, isotropic heat equation

\[ \frac{\partial u}{\partial t}(x,t) - \kappa \Delta u(x,t) = f(x), \quad x \in \Omega = \{ x : |x| < 1 \}, \quad 0 < t \leq T, \quad (1a) \]

with homogeneous Dirichlet boundary condition

\[ u(x,t) = 0, \quad x \in \partial \Omega, \quad 0 < t \leq T, \quad (1b) \]

initial condition

\[ u(x,0) = 0, \quad x \in \Omega, \quad (1c) \]

and source term \( f(x) = \exp\left(-\gamma \sqrt{x_1^2 + x_2^2}\right) \). Here, \( \kappa > 0 \) is the homogeneous, scalar conductivity, \( \gamma > 0 \) is some scaling factor, and \( T > 0 \) is the final time of observation.

The time derivative in (1a) is approximated using the finite differences \( \theta \)-method. Let us first define an equidistant grid \( 0 = t_0 < t_1 < \ldots < t_n = T \), with \( h_t = t_{i+1} - t_i \) for all \( 0 \leq i \leq n - 1 \). Then we set \( u^i \equiv u(\cdot, t_i) \) and define

\[ u^{i+\theta} \equiv (1 - \theta)u^i + \theta u^{i+1} \quad (2) \]

for \( 0 \leq \theta \leq 1 \). Finally, we approximate

\[ \frac{\partial u^{i+\theta}}{\partial t} \approx \frac{u^{i+1} - u^i}{h_t} \quad (3) \]

for \( 0 \leq i \leq n - 1 \).

The resulting scheme for \( \theta = 0 \) is the (standard or forward) Euler method, for \( \theta = 1/2 \) it is called Crank-Nicolson discretization and for \( \theta = 1 \) it is the backward Euler method.

a) Apply the finite differences \( \theta \)-method to (1) and transform (1) into weak form.

b) Show that the resulting variational formulation is an elliptic problem in every time step.
2. FE solver for transient, isotropic, homogeneous heat equation

a) Implement the finite differences $\theta$-method for the transient, isotropic, homogeneous heat equation (1). For the spatial discretization use your linear FE solver for elliptic boundary value problems.

b) Now we set $\theta = 1/2$. Choose some values for the conductivity $\kappa$ and the source term scaling $\gamma$. Furthermore, choose some time discretization $0 = t^0 < t^1 < \ldots < t^n = T$ and a FE mesh of $\Omega$. Then choose $T$ large enough such that your FE solution is close to the static limit, i.e., $|\frac{\partial u}{\partial t}(x,T)| \ll 1$.

c) Compare your time-dependent solution at $t = T$ with the solution of the corresponding stationary heat equation.

Give a 20 minutes talk on your results using black board and/or slides (pdf/ppt) on Thursday, July 17th, 2014, 10.15am, in MA 542. After the talk there will be 10 minutes discussion.

Coordinator: Dirk Klindworth, MA 665, 030/314-25192, klindworth@math.tu-berlin.de
Website: http://www.math.tu-berlin.de/?fem-lecture