Introduction to the Finite Element Method

Series 4

1. Discrete variational problems (Will be discussed on May 13th, 2014.)

Consider the variational problem: Find \( u \in V \subset L^2(]0,1[) \) such that

\[
\int_0^1 u(x) v(x) \, dx = \int_0^1 e^x v(x) \, dx
\]

for all \( v \in V \).

a) This variational problem is to be discretized in a Ritz-Galerkin fashion based on the following choices of trial/test spaces

(i) \( V_n = \text{span}\{x^k, k = 0, \ldots, n-1\} \),

(ii) \( V_n = \text{span}\{\sin(k\pi x), k = 1, \ldots, n\} \),

(iii) \( V_n = \text{span}\{\chi_{[k\frac{1}{n},(k+1)\frac{1}{n}]}, k = 1, \ldots, n\} \),

where \( n \in \mathbb{N} \) is the discretization parameter that serves as index for a family of trial/test spaces and \( \chi_I \) is the characteristic function of the interval \( I \), i.e.

\[
\chi_I(x) = \begin{cases} 
1 & \text{if } x \in I, \\
0 & \text{elsewhere.}
\end{cases}
\]

Compute the resulting linear systems of equations that correspond to the trial/test spaces (i) – (iii), when using the functions in the above definitions as basis functions.

b) Compute the condition numbers of the coefficient matrices of the linear systems of equations from a) for the schemes (ii) and (iii).

c) For the scheme (i) compute the condition number of the linear system of equations for \( n = 1, \ldots, 10 \) using of Matlab/Octave.

d) For the scheme (iii) compute the \( L^2(]0,1[) \)-norm of the discretization error as a function of the discretization parameter \( n \).

\textbf{Hint: Derive a formula for the discretization error as a function of the discretization parameter \( n \). You may evaluate this formula using Matlab/Octave and compute the rate of convergence using the command \texttt{diff(log(<error vector>))}/\texttt{diff(log(<n-vector>))}.}
2. **Mesh generation in Matlab** (Will be discussed on May 15th, 2014.)

Download the triangle mesh generator “mesh2d” (v24) from www.mathworks.com/matlabcentral/fileexchange/?term=mesh2d to some local directory. If you use Matlab version 7.14 (R2012a) or newer, the build-in function `tsearch` is not available anymore. Therefore, you have to do the following changes in the mesh2D source code:

- **mytesearch.m**, line 68: replace
  
  ```matlab
  i(j) = tsearch(x,y,t,xi(j),yi(j));
  ```
  
  by
  
  ```matlab
  i(j) = tsearchn([x y],t,[xi(j) yi(j)]);
  ```

- **meshfaces.m**, line 199: replace
  
  ```matlab
  i = tsearch(ph(:,1),ph(:,2),th,p(:,1),p(:,2));
  ```
  
  by
  
  ```matlab
  i = tsearchn(ph,th,p);
  ```

Finally, install the m-files of mesh2D by introducing the directory of the mesh generator to Matlab via `addpath()`.

- **a)** Write a function `meshSquare.m` which creates a mesh for a square where the side length and the maximal mesh width are input parameters. The function should return the coordinates of the nodes and the indices of the triangle nodes.

- **b)** Write a function `meshDisc.m` which creates a triangular mesh for a disc whose boundary is approximated by a polygon. The radius, the number of boundary nodes and the maximal mesh width are input parameters.

- **c)** Write a function `meshRing.m` which creates a mesh for a circular ring whose boundary is approximated by polygons. The inner radius, the outer radius, the number of boundary nodes and the maximal mesh width are input parameters.

- **d)** Are the the meshes in a), b) and c) meshes of the respective domain $\Omega$, which is a square, a disc or a circular ring?

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**Be prepared to present your results!** Exercise 1 will be discussed on May 13th, 2014, 10.15am, MA 545, and exercise 2 will be discussed on May 15th, 2014, 10.15am, MA 542.

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