TU Berlin — Department of Mathematics
Dr. Kersten Schmidt, Dirk Klindworth

winter term 2015/2016

NUMERICS OF PARTIAL DIFFERENTIAL EQUATIONS

Series 1

1. **Classification of second-order, linear partial differential equations**

   a) Using the definitions introduced in the lecture, show that the heat transfer equation is parabolic and that the wave equation is hyperbolic.

   b) Consider the second-order, linear partial differential equation

   $$-u_{xx} - yu_{yy} = f$$

   in $\Omega = (-1,1)^2$. Discuss the type of the equation for different $(x,y) \in \Omega$.

2. **Physical interpretation of Robin boundary conditions**
   Let us consider the homogeneous, isotropic heat transfer equation

   $$\partial_t u - \Delta u = 0$$

   with Robin boundary condition

   $$\nabla u \cdot \mathbf{n} + \alpha u = \beta,$$

   where $\mathbf{n}$ denotes the outward unit normal.

   Which sign needs α to have to be compatible with thermodynamics (and common sense)?

3. **Well-posedness of partial differential equations**
   Let $\Omega = (-1,1) \times (0,1)$. We consider the initial boundary value problem

   $$u_t - u_{xx} = 0, \qquad \text{in } \Omega, \tag{1a}$$
   $$u(\pm 1, t) = 0, \qquad \text{for } 0 \le t \le 1, \tag{1b}$$
   $$u(x,0) = \sin(n\pi x), \qquad \text{for } -1 \le x \le 1, n \in \mathbb{N}. \tag{1c}$$

   a) Give the analytical solution $u(x,t)$ for general $n \in \mathbb{N}$.
   Hint: Use separation of variables, i.e. $u(x,t) = u_1(x) \cdot u_2(t)$.

   b) Consider the problem: Find $u(x,1)$ for initial data $u(x,0) = \sin(n\pi x)$, $n \in \mathbb{N}$. Is this problem well posed with respect to the L^2-norm in $(-1,1)$?

   c) Replace $u_t - u_{xx} = 0$ in (1a) by $u_t + u_{xx} = 0$ and redo a) and b).

See next page!

4. **Time-harmonic Helmholtz equation in acoustics**
Let $\Omega \subset \mathbb{R}^d$. We consider the linear Euler equations

$$\begin{aligned}\partial_t \mathbf{v}(t,\mathbf{x}) + \mathbf{grad}\, p(t,\mathbf{x}) &= 0, \qquad t>0,\, \mathbf{x}\in\Omega,\\ \partial_t p(t,\mathbf{x}) + c^2 \operatorname{div} \mathbf{v}(t,\mathbf{x}) &= 0, \qquad t>0,\, \mathbf{x}\in\Omega,\end{aligned}$$

for the velocity $\mathbf{v}(t,\mathbf{x}) \in \mathbb{R}^d$ and the pressure $p(t,\mathbf{x}) \in \mathbb{R}$ in gases with the velocity of light c, and look for time-harmonic solutions $\hat{\mathbf{v}}(\mathbf{x}) \in \mathbb{C}^d$ and $\hat{p}(\mathbf{x}) \in \mathbb{C}$ with $\mathbf{v}(t,\mathbf{x}) = \operatorname{Re}\left(\hat{\mathbf{v}}(\mathbf{x})\mathrm{e}^{-\mathrm{i}\omega t}\right)$ and $p(t,\mathbf{x}) = \operatorname{Re}\left(\hat{p}(\mathbf{x})\mathrm{e}^{-\mathrm{i}\omega t}\right)$. Derive an equation for the time-harmonic pressure $\hat{p}$ only.

To be handed in by: October 27th, 2015 (10.15 a.m., before lecture starts)

This exercise series will be discussed in the tutorial class on October 29th, 2015, 2.15 p.m. in A 052.

Coordinator: Dirk Klindworth, MA 365, 030/314-25192, `klindworth@math.tu-berlin.de`
Website: `http://www.tu-berlin.de/?NumPDE`