TU Berlin — Department of Mathematics winter term 2015/2016
Dr. Kersten Schmidt, Dirk Klindworth

Numerics of partial differential equations

Series 2

1. For $\Omega = (0, 1)$ give an example of a function $u \in C^1(\Omega)$ that does not possess a weak gradient bounded in $L^2(\Omega)$.

2. Show that for $\mathbf{f} \in (C^1(\Omega))^d \cap (C^0(\overline{\Omega}))^d$

$$\int_\Omega \mathbf{curl}\, \mathbf{f} \,\mathrm{d}\boldsymbol{\xi} = \int_\Gamma \mathbf{n} \times \mathbf{f} \,\mathrm{d}S$$

Hint: Use

$$\int_\Omega \langle \mathbf{curl}\, \mathbf{u}, \mathbf{f} \rangle - \langle \mathbf{u}, \mathbf{curl}\, \mathbf{f} \rangle \,\mathrm{d}\boldsymbol{\xi} = \int_\Gamma \langle \mathbf{u} \times \mathbf{f}, \mathbf{n} \rangle \,\mathrm{d}S$$

for $\mathbf{u} = \mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3$ *the canonical basis of* $\mathbb{R}^3$.

3. Let $u \in L^2(\Omega)$ and $\boldsymbol{\alpha} \in \mathbb{N}_0^n$. A function $w \in L^2(\Omega)$ is called the *weak derivative* or *distributional derivative* $\partial^{\boldsymbol{\alpha}} u$ (of order $|\boldsymbol{\alpha}|$) of u, if

$$\int_\Omega w v \,\mathrm{d}\boldsymbol{\xi} = (-1)^{|\boldsymbol{\alpha}|} \int_\Omega u \,\partial^{\boldsymbol{\alpha}} v \,\mathrm{d}\boldsymbol{\xi} \qquad \forall v \in C_0^\infty(\Omega)\ .$$

Based on this definition, all linear differential operators can be given a weak/distributional interpretation. For example, the *weak gradient* $\mathbf{grad}\, u$ of a function $u \in L^2(\Omega)$ is a vector field $\mathbf{w} \in (L^2(\Omega))^d$ with

$$\int_\Omega \langle \mathbf{w}, \mathbf{v} \rangle \,\mathrm{d}\boldsymbol{\xi} = -\int_\Omega u \;\mathrm{div}\, \mathbf{v} \,\mathrm{d}\boldsymbol{\xi} \qquad \forall \mathbf{v} \in (C_0^\infty(\Omega))^d\ .$$

Provide a definition of the *weak rotation* $\mathbf{curl}\, \mathbf{u} \in (L^2(\Omega))^3$.

4. Let $\Omega \subset \mathbb{R}^d$ with Lipschitz continuous boundary $\partial\Omega$. We consider the boundary value problems

$$-\,\mathrm{div}\, \sigma\, \mathbf{grad}\, u + cu = f \qquad \text{in } \Omega, \tag{1a}$$
$$u = g \qquad \text{on } \partial\Omega, \tag{1b}$$

and

$$-\,\mathrm{div}\, \sigma\, \mathbf{grad}\, u + cu = f \qquad \text{in } \Omega, \tag{2a}$$
$$\sigma\, \mathbf{grad}\, u \cdot \mathbf{n} + hu = g \qquad \text{on } \partial\Omega. \tag{2b}$$

Choose appropriate test function spaces V, i.e. $V = C_0^\infty(\Omega)$ or $V = C^\infty(\Omega) \cap C^0(\overline{\Omega})$, and give the variational formulations of (1) and (2). Point out whether the boundary conditions (1b) and (2b) are *natural* and can be incorporated directly into the variational formulations, or *essential* and need to be incorporated into the trial function spaces U.

See next page!

To be handed in by: October 27th, 2015 (10.15 a.m., before lecture starts)

This exercise series will be discussed in the tutorial class on October 29th, 2015, 2.15 p.m. in A 052.

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