TU Berlin — Department of Mathematics
Dr. Kersten Schmidt, Dirk Klindworth winter term 2015/2016

NUMERICS OF PARTIAL DIFFERENTIAL EQUATIONS

Series 3

1. Can the Sobolev space $H^1(\Omega)$ be regarded as a closed subspace of $L^2(\Omega)$? Give an explanation!

2. Consider the boundary value problem

$$-\operatorname{div}\sigma\,\mathbf{grad}\,u + cu = f \qquad \text{in } \Omega, \tag{1a}$$

$$u = 0 \qquad \text{on } \partial\Omega. \tag{1b}$$

a) Give the variational formulation of (1).

b) What is the corresponding energy norm? Which conditions need σ and c to fulfill such that the energy norm is well defined?

c) What is the "ideal" Sobolev space V for the variational formulation of (1)?

d) Show that the bilinear form b in the variational formulation is V-elliptic and continuous in the Sobolev space V.

e) Let $\Omega_l \subset \Omega$, $l = 1, \ldots, L$, with $\overline{\Omega} = \bigcup_{l=1}^{L} \overline{\Omega}_l$ and $\Omega_k \cap \Omega_l = \emptyset$ for $k \neq l$. Now we consider a subspace of V of functions v whose restriction $v|_{\Omega_l}$ can be extended to a continuously differentiable function in $\overline{\Omega}_l$ for all $l = 1, \ldots, L$. Discuss the global continuity of these functions v.

f) Suppose the solution $u \in V$ of (1) has higher regularity in Ω_l for all $l = 1, \ldots, L$, *i. e.* at least $u \in C^2(\Omega_l)$. Which additional continuity does u have on the inner boundaries $\partial\Omega_l \setminus \partial\Omega$ for all $l = 1, \ldots, L$?

3. A low frequency approximation to linear Maxwell's equations describing the alternating electric and magnetic fields $\mathbf{E}$ and $\mathbf{H}$ in a domain $\Omega \subset \mathbb{R}^3$ is the time-harmonic eddy current model (Wirbelstrommodell)

$$\mathbf{curl}\,\mathbf{E} = \mathrm{i}\omega\mu\mathbf{H}, \tag{2a}$$

$$\mathbf{curl}\,\mathbf{H} = \sigma\mathbf{E}, \tag{2b}$$

where $\omega > 0$ is the (angular) frequency, $\mu \geq \mu_0 > 0$ the magnetic permeability and $\sigma \geq \sigma_0 > 0$ the electric conductivity. On the boundary $\partial\Omega$ of the domain Ω the tangential component of the electric field is prescribed as

$$\mathbf{E} \times \mathbf{n} = \mathbf{E}_{0,T}. \tag{2c}$$

a) Derive a variational formulation of (2) by taking the weak form of (2a) with test function $\mathbf{H}'$, taking the strong form of (2b) multiplied by a test function $\mathbf{E}' = \sigma^{-1}\,\mathbf{curl}\,\mathbf{H}'$.

b) Is the boundary condition (2c) essential or natural?

c) What is the "ideal" Sobolev space V for the variational formulation of (2)?

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d) Let $\Omega_l \subset \Omega$, $l = 1, \ldots, L$, with $\overline{\Omega} = \bigcup_{l=1}^{L} \overline{\Omega}_l$ and $\Omega_k \cap \Omega_l = \emptyset$ for $k \neq l$. Now we consider a subspace of V of functions $\mathbf{v}$ whose restriction $\mathbf{v}|_{\Omega_l}$ can be extended to a continuously differentiable function in $\overline{\Omega}_l$ for all $l = 1, \ldots, L$. Discuss the global continuity of these functions $\mathbf{v}$.

e) If V is a complex vector space, which is the case for the variational formulation of (2), a bilinear form b is called V-elliptic if there exists a constant $\theta \in [0, 2\pi)$ such that

$$\exists \gamma > 0 : \quad \mathrm{Re}\left(e^{\mathrm{i}\theta}\ \mathsf{b}(v, \overline{v})\right) \geq \gamma \|v\|_V^2 \qquad \forall v \in V,$$

where $\overline{v}$ is the complex conjugate of v.
Now let $\sigma \equiv \mu \equiv \omega = 1$. Show that the bilinear form of the variational formulation of (2) is V-elliptic.

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4. Programming exercise: Discrete variational problems (due to November 10th, 2015, 10.15 a.m.)*

Consider the variational problem: Find $u \in V \subset L^2(]0,1[)$ such that

$$\int_0^1 u(x)\, v(x)\, \mathrm{d}x = \int_0^1 \mathrm{e}^x\, v(x)\, \mathrm{d}x \tag{3}$$

for all $v \in V$.

a) This variational problem is to be discretized in a Ritz-Galerkin fashion based on the following choices of trial/test spaces

(i) $V_n = \operatorname{span}\{x^k,\, k = 0, \dots, n-1\}$,

(ii) $V_n = \operatorname{span}\{\sin(k\pi x),\, k = 1, \dots, n\}$,

(iii) $V_n = \operatorname{span}\{\chi_{[\frac{k-1}{n}, \frac{k}{n}[},\, k = 1, \dots, n\}$,

where $n \in \mathbb{N}$ is the **discretization parameter** that serves as index for a family of trial/test spaces and χ_I is the characteristic function of the interval I, i.e.

$$\chi_I(x) = \begin{cases} 1 & \text{if } x \in I, \\ 0 & \text{elsewhere.} \end{cases}$$

Compute the resulting linear systems of equations that correspond to the trial/test spaces (i) – (iii), when using the functions in the above definitions as basis functions of V_n and inserting any combination of these basis functions in (3) for u and v.

b) Compute the condition numbers of the coefficient matrices of the linear systems of the schemes (ii) and (iii).

c) For the scheme (i) compute the condition number of the linear system of equations for $n = 1, \dots, 10$ using Python.

d) Plot the condition numbers of the schemes (i)–(iii) in dependence on the degrees of freedom $n = 1, \dots, 10$ in one figure using Python and Matplotlib.

e) For the scheme (iii) compute the $L^2(]0,1[)$-norm of the discretization error as a function of the discretization parameter n using Python.
Hint: Derive a formula for the discretization error as a function of the discretization parameter n. Evaluate this formula using Python and compute the rate of convergence using the Numpy functions `diff` *and* `log` *and executing* `diff(log(<error vector>))/diff(log(<n-vector>))`.

To be handed in by: November 3rd, 2015, 10.15 a.m. (before lecture starts)
Please hand in your solutions of at least two exercises of the three theoretical exercises 1.–3.

* Send a Python script to `klindworth@math.tu-berlin.de` that executes all steps of 4.d) and 4.e)!

This exercise series will be discussed in the tutorial class on November 5th, 2015, 2.15 p.m. in A 052.

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