TU Berlin — Department of Mathematics
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# Numerics of partial differential equations

## Series 4

**1.** Let $V$ be a reflexive Banach space. A form $\mathsf{a}(\cdot,\cdot) : V \times V \to \mathbb{C}$ is called sesquilinear if

$$\mathsf{a}(u_1 + \lambda u_2, v) = \mathsf{a}(u_1, v) + \lambda \mathsf{a}(u_2, v)$$
$$\mathsf{a}(u, v_1 + \lambda v_2) = \mathsf{a}(u, v_1) + \bar{\lambda} \mathsf{a}(u, v_2),$$

for all $u_1, u_2, v_1, v_2 \in V$ and $\lambda \in \mathbb{C}$, and a sesquilinear form $\mathsf{a}$ is called $V$-elliptic if

$$\exists\, \gamma > 0,\, \theta \in [0, 2\pi[ \quad \text{such that} \quad \mathrm{Re}\left(\mathrm{e}^{\mathrm{i}\theta}\, \mathsf{a}(v,v)\right) \geq \gamma \|v\|_V^2 \qquad \forall v \in V.$$

Now let $\mathsf{b}(\cdot,\cdot) : V \times V \to \mathbb{C}$ be a sesquilinear form that is $V$-elliptic. Show that $\mathsf{b}$ fulfills the inf-sup conditions

$$\exists\, \gamma > 0 \quad \text{such that} \quad \inf_{w \in U \setminus \{0\}} \sup_{v \in V \setminus \{0\}} \frac{|\mathsf{b}(w,v)|}{\|w\|_U \|v\|_V} \geq \gamma \tag{IS1}$$

and

$$\sup_{w \in U \setminus \{0\}} |\mathsf{b}(w,v)| > 0 \qquad \forall v \in V \setminus \{0\}, \tag{IS2}$$

with $U = V$.

**2.** Let $U$ and $V$ be reflexive Banach spaces. Consider the linear variational problem: find $u \in U$ such that

$$\mathsf{b}(u,v) = \langle f, v \rangle_{V' \times V} \qquad \forall v \in V. \tag{LVP}$$

Show the following statements:

a) If (IS1) does not hold then the solution $u$ of the linear variational problem (LVP) is not unique.

b) If (IS2) does not hold then a solution of the linear variational problem (LVP) does not necessarily exist.

**3.** On the Banach space $C^0([0,1])$, equipped with the supremum norm, consider the bilinear form

$$\mathsf{b}(u,v) := \int_0^1 u(x) v(x)\, \mathrm{d}x.$$

a) Show that the inf-sup condition (IS1) is not satisfied.

b) In which space does the bilinear form $\mathsf{b}$ satisfy (IS1) and (IS2)?

**See next page!**

**4.** Show that a variational problem based on the bilinear form

$$\mathsf{b}(u,v) := \int_0^{2\pi}\int_0^{2\pi} \mathrm{e}^{x-y} u(x) v(y) \,\mathrm{d}x \mathrm{d}y$$

does in general fail to possess a unique solution in the Banach space $L^2(]0,2\pi[)$.

**To be handed in by:** November 10th, 2015 (10.15 a.m., before lecture starts)

This exercise series will be discussed in the tutorial class on November 12th, 2015, 2.15 p.m. in A 052.

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