TU Berlin — Department of Mathematics winter term 2015/2016
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NUMERICS OF PARTIAL DIFFERENTIAL EQUATIONS

Series 6

1. Let $\Omega = \{\mathbf{x} \in \mathbb{R}^2 \text{ such that } |\mathbf{x}| < 1 \text{ and } x_2 > 0\}$. We consider the boundary segment $\Gamma = \{\mathbf{x} \in \partial\Omega \text{ such that } x_2 = 0\}$. Recall that the Sobolev-Slobodeckij-norm of a function $v \in H^{1/2}(\Gamma)$ reads

$$\|v\|^2_{H^{1/2}(\Gamma)} := \int_\Gamma \int_\Gamma \frac{|v(\mathbf{x}) - v(\mathbf{y})|^2}{|\mathbf{x} - \mathbf{y}|^2} \, \mathrm{d}S(\mathbf{x})\mathrm{d}S(\mathbf{y}).$$

a) Show that the function

$$g_1 : \Gamma \longmapsto \mathbb{R}, \quad g_1(\mathbf{x}) := \begin{cases} 1, & \text{if } x_1 > 0 \,, \\ 0, & \text{if } x_1 < 0 \,, \end{cases}$$

does not belong to $H^{1/2}(\Gamma)$.

b) Show that the function

$$g_2 : \Gamma \longmapsto \mathbb{R}, \quad g_2(\mathbf{x}) := \sqrt{|\mathbf{x}|}$$

belongs to $H^{1/2}(\Gamma)$ but not to $H^1(\Gamma)$.
Hint: Instead of showing that $\|g_2\|_{H^{1/2}(\Gamma)}$ is bounded you may use the fact that g_2 is the restriction of $h : \Omega \to \mathbb{R},\ \mathbf{x} \mapsto \sqrt{|\mathbf{x}|}$ to Γ and prove that $h \in H^1(\Omega)$.

2. Let $\Omega \subset \mathbb{R}^d$ be a bounded open Lipschitz domain. Give a Sobolev space V for which the trace operator $\mathsf{R}\, u = \mathbf{grad}\, u \cdot \mathbf{n}$ defines a continuous mapping from V onto $H^{-1/2}(\partial\Omega)$.

3. Let W, V be reflexive Banach spaces, and $W_n \subset W$, $V_n \subset V$ discrete subspaces. Let $\mathsf{b} : W \times V \to \mathbb{R}$ be a bounded bilinear form satisfying the inf-sup conditions (IS1) and (IS2), and $f \in V'$. Furthermore, let $u \in W$ solve $\mathsf{b}(u, v) = \langle f, v\rangle_{V' \times V}$ for all $v \in V$, and let $u_n \in W_n$ solve $\mathsf{b}(u_n, v_n) = \langle f, v_n\rangle_{V' \times V}$ for all $v_n \in V_n$.

Prove the uniqueness of u_n and the stability estimate (2.20) in Theorem 2.29, i.e. show that u_n is unique and satisfies

$$\|u_n\|_W \leq \frac{1}{\gamma_n} \|f\|_{V_n'} = \frac{1}{\gamma_n} \sup_{v_n \in V_n} \frac{|f(v_n)|}{\|v_n\|_V},$$

if b satisfies the discrete inf-sup condition

$$\exists \gamma_n > 0 : \qquad \inf_{w_n \in W_n \setminus \{0\}} \sup_{v_n \in V_n \setminus \{0\}} \frac{|\mathsf{b}(w_n, v_n)|}{\|w_n\|_W \, \|v_n\|_V} \geq \gamma_n. \tag{DIS}$$

See next page!

4. Let V be a reflexive Banach space, and $V_n \subset V$ a discrete subspace. Let $\mathsf{b} : V \times V \to \mathbb{R}$ be a bounded bilinear form and $f \in V'$. Furthermore, let $u \in V$ solve $\mathsf{b}(u, v) = \langle f, v \rangle_{V' \times V}$ for all $v \in V$, and let $u_n \in V_n$ solve $\mathsf{b}(u_n, v_n) = \langle f, v_n \rangle_{V' \times V}$ for all $v_n \in V_n$.

Prove Céa's lemma, i.e. show that

$$\|u - u_n\|_V \leq \frac{\|\mathsf{b}\|_{V \times V \mapsto \mathbb{R}}}{\gamma_e} \inf_{v_n \in V_n} \|u - v_n\|_V$$

if the bilinear form b is V-elliptic with ellipticity constant γ_{e}, cf. Theorem 2.31!

To be handed in by: November 24th, 2015, 10.15 a.m. (before lecture starts)

This exercise series will be discussed in the tutorial class on November 26th, 2015, 2.15 p.m. in A 052.

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