NUMERICS OF PARTIAL DIFFERENTIAL EQUATIONS

Series 6

1. Let $\Omega = \{ \mathbf{x} \in \mathbb{R}^2 \text{ such that } |\mathbf{x}| < 1 \text{ and } x_2 > 0 \}$. We consider the boundary segment $\Gamma = \{ \mathbf{x} \in \partial \Omega \text{ such that } x_2 = 0 \}$. Recall that the Sobolev-Slobodeckij-norm of a function $v \in H^{1/2}(\Gamma)$ reads

$$||v||_{H^{1/2}(\Gamma)}^2 := \int_{\Gamma} \int_{\Gamma} \frac{|v(\mathbf{x}) - v(\mathbf{y})|^2}{|\mathbf{x} - \mathbf{y}|^2} dS(\mathbf{x}) dS(\mathbf{y}).$$

a) Show that the function

$$g_1: \Gamma \longmapsto \mathbb{R}, \quad g_1(\mathbf{x}) := \begin{cases} 1, & \text{if } x_1 > 0, \\ 0, & \text{if } x_1 < 0, \end{cases}$$

does not belong to $H^{1/2}(\Gamma)$.

b) Show that the function

$$g_2: \Gamma \longmapsto \mathbb{R}, \quad g_2(\mathbf{x}) := \sqrt{|\mathbf{x}|}$$

belongs to $H^{1/2}(\Gamma)$ but not to $H^1(\Gamma)$.

Hint: Instead of showing that $||g_2||_{H^{1/2}(\Gamma)}$ is bounded you may use the fact that g_2 is the restriction of $h: \Omega \to \mathbb{R}$, $\mathbf{x} \mapsto \sqrt{|\mathbf{x}|}$ to Γ and prove that $h \in H^1(\Omega)$.

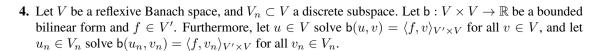
- **2.** Let $\Omega \subset \mathbb{R}^d$ be a bounded open Lipschitz domain. Give a Sobolev space V for which the trace operator $\mathsf{R}\,u = \mathbf{grad}\,u \cdot \mathbf{n}$ defines a continuous mapping from V onto $H^{-1/2}(\partial\Omega)$.
- 3. Let W,V be reflexive Banach spaces, and $W_n\subset W,V_n\subset V$ discrete subspaces. Let $\mathbf{b}:W\times V\to\mathbb{R}$ be a bounded bilinear form satisfying the inf-sup conditions (IS1) and (IS2), and $f\in V'$. Furthermore, let $u\in W$ solve $\mathbf{b}(u,v)=\langle f,v\rangle_{V'\times V}$ for all $v\in V$, and let $u_n\in W_n$ solve $\mathbf{b}(u_n,v_n)=\langle f,v_n\rangle_{V'\times V}$ for all $v\in V_n$.

Prove the uniqueness of u_n and the stability estimate (2.20) in Theorem 2.29, i.e. show that u_n is unique and satisfies

$$||u_n||_W \le \frac{1}{\gamma_n} ||f||_{V_n'} = \frac{1}{\gamma_n} \sup_{v_n \in V_n} \frac{|f(v_n)|}{||v_n||_V},$$

if b satisfies the discrete inf-sup condition

$$\exists \gamma_n > 0: \quad \inf_{w_n \in W_n \setminus \{0\}} \sup_{v_n \in V_n \setminus \{0\}} \frac{|\mathsf{b}(w_n, v_n)|}{\|w_n\|_W \|v_n\|_V} \ge \gamma_n. \tag{DIS}$$



Prove Céa's lemma, i.e. show that

$$\|u - u_n\|_V \le \frac{\|\mathbf{b}\|_{V \times V \to \mathbb{R}}}{\gamma_e} \inf_{v_n \in V_n} \|u - v_n\|_V$$

if the bilinear form b is V-elliptic with ellipticity constant $\gamma_{\rm e}$, cf. Theorem 2.31!

To be handed in by: November 24th, 2015, 10.15 a.m. (before lecture starts)

This exercise series will be discussed in the tutorial class on November 26th, 2015, 2.15 p.m. in A 052.

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